MAPS BETWEEN SYMMETRIC POWERS AND APPLICATIONS TO SYMMETRIC GROUPS

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1. Symmetric powers

1.1. **Definition.** Let K be a field, G be a group and V be a finite-dimensional KG-module. The r-fold tensor product $V^{\otimes r}$ is made into a KG-module via the diagonal action

$$g \cdot (v_1 \otimes \cdots \otimes v_r) = (gv_1) \otimes \cdots \otimes (gv_r).$$

In general $v \otimes w \neq w \otimes v$; the symmetric powers change this. The *r-th symmetric power of V* is defined as

$$\operatorname{Sym}^r V = V^{\otimes r} / \langle v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_r - v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_r \rangle$$

(where the 'denominator' can be considered as spanned by the vectors as vector space or generated by the generators as a submodule).

Alternatively, $\operatorname{Sym}^r V$ can be thought of as homogeneous polynomial of degree r in vectors of V (i.e. forget $\otimes$). In particular if $x_1, \ldots, x_t$ is a basis of V, then there is an isomorphism $\operatorname{Sym}^r V \cong K[x_1, \ldots, x_t]$, where G acts on the latter by

$$g \cdot (x_1^{\alpha_1} \cdots x_t^{\alpha_t}) = (gx_1)^{\alpha_1} \cdots (gx_t)^{\alpha_t}.$$

Example 1.1. If $g \in G$ acts on basis x, y of a two-dimensional KG-module by $gx = 2x - y$ and $gy = -x$, then $g \cdot x^2 y = (2x - y)^2(-x) = -4x^3 + 4x^2 y - xy^2$.

Symmetric powers are central object of invariant theory; invariants are generators of trivial submodules of symmetric powers. They also for an example of more general endofunctor: *polynomial Schur functor* ∇^λ. These polynomial Schur functors give rise to all irreducible complex polynomial representations of $\mathrm{GL}_t(\mathbb{C})$.

1.2. **Main result.** We abuse the notation and write K for the trivial KG-module. Given a short exact sequence

$$0 \to W \to V \to K \to 0 \tag{1}$$

of KG-modules one obtains for any $r \geq 1$ a short exact sequence

$$0 \to \operatorname{Sym}^r W \to \operatorname{Sym}^r V \to \operatorname{Sym}^{r-1} V \to 0. \tag{2}$$

'Dually', given a short exact sequence

$$0 \to K \to V \to W \to 0, \tag{3}$$

for any $r \geq 1$ there is a short exact sequence

$$0 \to \operatorname{Sym}^{r-1} V \to \operatorname{Sym}^r V \to \operatorname{Sym}^r W \to 0. \tag{4}$$

Date: October 1, 2025.

Recall that a short exact sequence

$$0 \to L \xrightarrow{\phi} M \xrightarrow{\theta} N \to 0$$

splits if there is a map $\iota : N \to M$ such that $\theta \circ \iota = \mathrm{id}_N$, or equivalently, if there is a map $\psi : M \to L$ such that $\psi \circ \phi = \mathrm{id}_L$. If that happens, then $M \cong L \otimes N$.

Theorem 1.2. *Let K be a field of prime characteristic p, V be a KG-module and r_0 be a positive integer which is not congruent to 0 or -1 modulo p.*

(1) Suppose that V has the trivial module as a quotient (i.e. there is a short exact sequence (1)*). If* (2) *splits for $r = r_0$, then is splits for $r = r_0 + 1$.*

(2) Suppose that V has the trivial module as a submodule (i.e. there is a short exact sequence (3)*). If* (4) *splits for $r = r_0$, then is splits for $r = r_0 + 1$.*

We demonstrate application of the theorem on a result about permutation modules. A KG-module is called *a permutation module* if it has G-invariant basis (i.e. G permutes this basis).

Corollary 1.3. *Let K be a field of prime characteristic p and V be a permutation KG-module. Then for any $2 \leq r \leq p-1$, the symmetric power $\mathrm{Sym}^{r-1} V$ is a summand of $\mathrm{Sym}^r V$.*

Proof. Let $x_1, \dots x_t$ be a permutation basis of V. Then the map $\theta : V \to K$ sending each x_i to 1 is a surjective KG-homomorphism, yielding a short exact sequence of the form (1). We prove that the short exact sequence (2) splits for $2 \leq r \leq p-1$.

One checks that the surjection $\theta_2 : \mathrm{Sym}^2 V \to V$ in (2) with $r = 2$ sends $x_i x_j$ to $x_i + x_j$. We find out that this short exact sequence split: the map witnessing this is $\iota : V \to \mathrm{Sym}^2 V$ given by $x_i \mapsto x_i^2/2$.

If the statement holds for $2 \leq r \leq p-2$, then it also holds for $r+1$ by part (i) of the main theorem. This finishes the proof by induction from which we immediately deduce the desired statement. □

1.3. **Map construction.** We focus on the first part of the main theorem. The idea is to construct a new splitting $\iota' : \mathrm{Sym}^r V \to \mathrm{Sym}^{r+1} V$ from a splitting $\iota : \mathrm{Sym}^{r-1} V \to \mathrm{Sym}^r V$ for r. To do so one uses a more general construction: given $\phi : \mathrm{Sym}^a V \to \mathrm{Sym}^b V$, we can construct for any $d \geq a-b$ non-negative a map $\Psi_d(\phi) : \mathrm{Sym}^d V \to \mathrm{Sym}^{d+b-a} V$ such that $\Psi_a(\phi) = \phi$ as composition of

$$\mathrm{Sym}^d V \xrightarrow{\text{split}} \mathrm{Sym}^a V \otimes \mathrm{Sym}^{d-a} V \xrightarrow{\phi\otimes\mathrm{id}} \mathrm{Sym}^b V \otimes \mathrm{Sym}^{d-a} V \xrightarrow{\text{mult}} \mathrm{Sym}^{d+b-a} V.$$

Example 1.4. Suppose that $\phi : V \to \mathrm{Sym}^3 V$ sends x to $x^2 y$ and y to $2x^3$. Then $\Psi_3(\phi)(xy^2) = 2xy\phi(y) + y^2\phi(x) = 4x^4 y + x^2 y^3$.

Example 1.5. If ϕ is the identity map then $\Phi_d(\phi)$ is scalar multiplication by $\binom{d}{a}$.

More generally, there is a practical formula for $\Psi_d(\phi)$.

Lemma 1.6. *Let $\phi : \mathrm{Sym}^a V \to \mathrm{Sym}^b V$ be a homomorphism of KG-modules. After the identification $\mathrm{Sym}\, V \cong K[x_1, \dots, x_t]$, we obtain that*

$$\Psi_d(\phi) = \sum_{\alpha_1+\cdots+\alpha_t=a} \phi(x_1^{\alpha_1} \cdots x_t^{\alpha_t}) \frac{\partial^a}{\alpha_1! \cdots \alpha_t! \partial x_1^{\alpha_1} \cdots \partial x_t^{\alpha_t}}.$$

Remark 1.7. One should worry about dividing by 0, but that is easily fixable.

With this construction, one can do some algebra and discover a formula for lifting a splitting.

2. Application to symmetric groups

2.1. **Background.** Let S_n be the symmetric group indexed by n and $M = M^{(n-1,1)}$ its natural permutation module with permutation basis $x_1, \dots, x_n$. The Specht module $S = S^{(n-1,1)}$ is the submodule of M of codimension 1 spanned by vectors $x_i - x_j$. We obtain a short exact sequence

$$0 \to S \to M \to K \to 0.$$

Over a field of characteristic 0 or $p \nmid n$, the Specht module S is irreducible. However, if $p \mid n$, then S has a trivial submodule spanned by $x_1 + \cdots + x_n$ giving rise to a short exact sequence

$$0 \to K \to S \to D \to 0,$$

with $D = D^{(n-1,1)}$ irreducible.

Both S and D give rise to further Specht (=irreducible in characteristic 0) and irreducible modules via their exterior powers, which are a common focus of research. For instance, the vertices of these modules are known in majority of the cases due to several authors, namely Danz, Giannelli, Lim, Müller, Wildon, and Zimmermann. When moving to symmetric powers, the modules stop being Specht/irreducible (or even indecomposable), and they grow in size, making them more difficult to study compared to the exterior counterparts. However, we can use the main result to understand them better.

2.2. **Symmetric powers of S and D.** Using the main theorem and a suitably chosen initial splitting we obtain.

Proposition 2.1. *Suppose that $p \mid n$.*

1. *The short exact sequence* (2) *with $V = M$ and $W = S$ splits for $2 \le r \le p-1$.*
2. *The short exact sequence* (4) *with $V = S$ and $W = D$ splits for $3 \le r \le p-1$.*

Remark 2.2. The first assertion is basically Corollary 1.3.

In turn, we obtain isomorphisms

$$\mathrm{Sym}^r S \oplus \mathrm{Sym}^{r-1} M \cong \mathrm{Sym}^r M$$

for $2 \leq r \leq p-1$, and

$$\mathrm{Sym}^r D \oplus 2\,\mathrm{Sym}^{r-1} M \cong \mathrm{Sym}^r M \oplus \mathrm{Sym}^{r-2} M$$

for $3 \leq r \leq p-1$.

The strength of these isomorphisms is that the symmetric powers of M are much easier to study than the symmetric powers of S and D (they can be written as direct sum of other permutation modules, which further decompose as direct sums of indecomposable Young modules). One deduces the following.

Theorem 2.3. *Suppose that K is a field of characteristic p such that $p \mid n \geq 3$. The symmetric power $\mathrm{Sym}^r S$ has a Specht filtration for $0 \leq r \leq p-1$ and the symmetric power $\mathrm{Sym}^r D$ has a Specht filtration for $2 \leq r \leq p-1$.*

Remark 2.4. The case $r = 2$ for the latter symmetric power must be done directly.

Complementing the results for exterior powers of S and D we have also found vertices of the indecomposable summands of our symmetric powers.

Recall that a *vertex* of a KG-module V is a minimal subgroup $H \leq G$ for which there is an indecomposable KH-module U, called a *source* of V, such that V is an indecomposable summand of the induced module $U\big\uparrow_H^G$. If K has characteristic p, then all vertices of V are p-groups.

A motto one should have in mind is 'smaller vertex means better-behaved module'. In particular, module is projective if and only if its vertex is the trivial group.

Theorem 2.5. *Suppose that K is a field of characteristic p such that $p \mid n \geq 3$. The indecomposable summands of the symmetric powers $\mathrm{Sym}^r S^{(n-1,1)}$ with $2 \leq r \leq p-1$ and of the symmetric powers $\mathrm{Sym}^r D^{(n-1,1)}$ with $3 \leq r \leq p-1$ have a trivial source and have a vertex given by a Sylow p-subgroup of S_{n-p} or S_{n-2p}.*

Remark 2.6. The statement does not hold for smaller r. This is easy to see using the fact that the p-valuation of the index of the vertex in S_n is less than the p-valuation of the dimension of the module. This also shows that the vertices in the statements are the least maximal possible candidates.

The isomorphism above can be used to compute new p-Kostka numbers (which describe decomposition of the permutation modules into indecomposable summands). (We get some inequalities when writing everything out).