The Mullineux map and partitions with divisible arms

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Partitions

Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ of n is a non-increasing sequence of positive integers, called **parts**, which add up to n. One way to represent a partition λ is using its **Young diagram** $Y(\lambda) = \{(i,j) \in \mathbb{N}^2 : i \le t, j \le \lambda_i\}$.

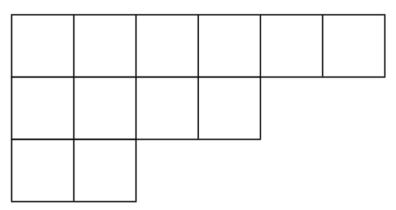

Fig.1: The Young diagram of partition $(6,4,2)$.

Alternatively, partitions can be visualised using the **James abacus** with 1 or $e > 1$ runners.

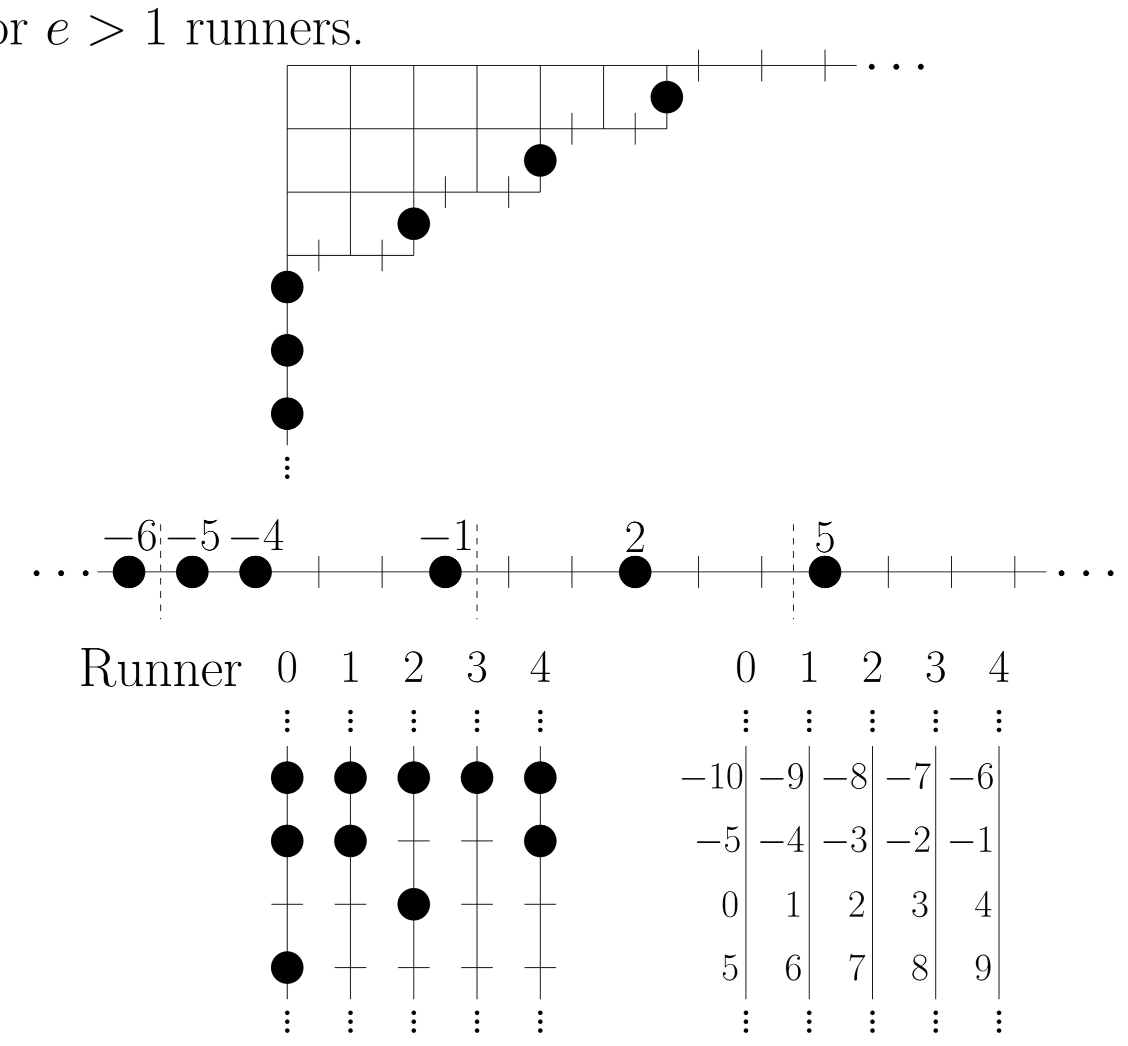

Fig. 2: The creation of the James abacus with 1 runner (in the middle) from a Young diagram. By cutting the runner into intervals of length $e = 5$ one obtains the James abacus with 5 runners (on the bottom left). Its positions are naturally ordered; see the bottom right diagram.

One recovers the corresponding partition from its James abacus by counting the empty spaces before each bead (ignoring any zeros).

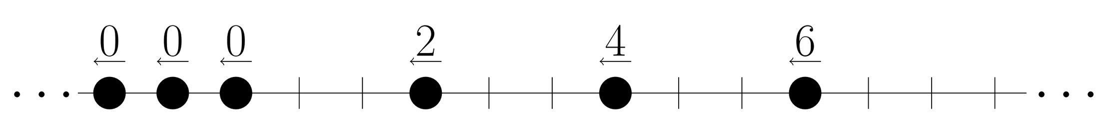

Fig. 3: Ignoring the zeros, we read off partition $(6,4,2)$.

A connected removable group of ae boxes (for some $a > 0$) with no 2×2 square is called an **e-hook**. Its **arm-length** equals the number of columns the e-hook spans minus one. The **e-core** of a partition is obtained by a consecutive removal of e-hooks.

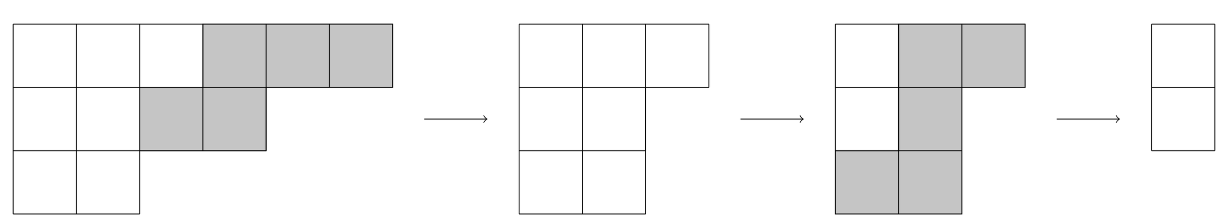

Fig. 4: The 5-core of $(6,4,2)$ is $(1,1)$. The grey 5-hooks have arm lengths 3 and 2, respectively.

Mullineux map

Fix integer e. For e prime, the irreducible modules D^λ of the symmetric group S_n in characteristic e are labelled by **e-regular** (= no e equal parts) partitions of n. The **Mullineux map** m_e sends an e-regular partition λ to an e-regular partition μ such that

$$D^\mu \cong D^\lambda \otimes \operatorname{sgn}.$$

The definition can be generalised to any $e \ge 2$, using an involution of the Hecke algebras of type A.

We will focus on the map m'_e, the composition of m_e with **partition conjugation** (= 'transposition'), arising from the decomposition matrices; see Figure 5.

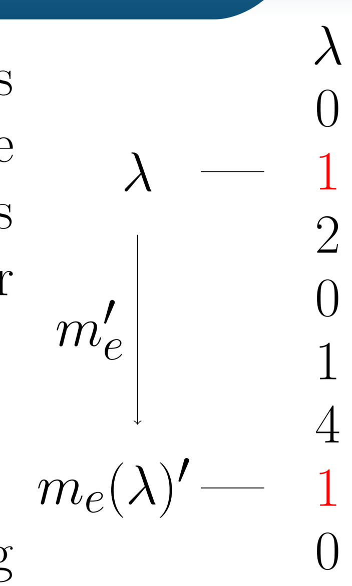

Fig. 5: A column of a decomposition matrix labelled by partition λ. Its first and last non-zero entries are highlighted.

Abacus Mullineux algorithm

We define the **leading beads** $b_0, b_1, \ldots$ of an abacus by: b_0 is the greatest bead and b_i is the greatest bead less or equal to $b_{i-1} - e$ $(i \ge 1)$. The **new bead** equals $b_i + ie$ (for any/all sufficiently large i). See the top left and the second right abaci for examples, respectively.

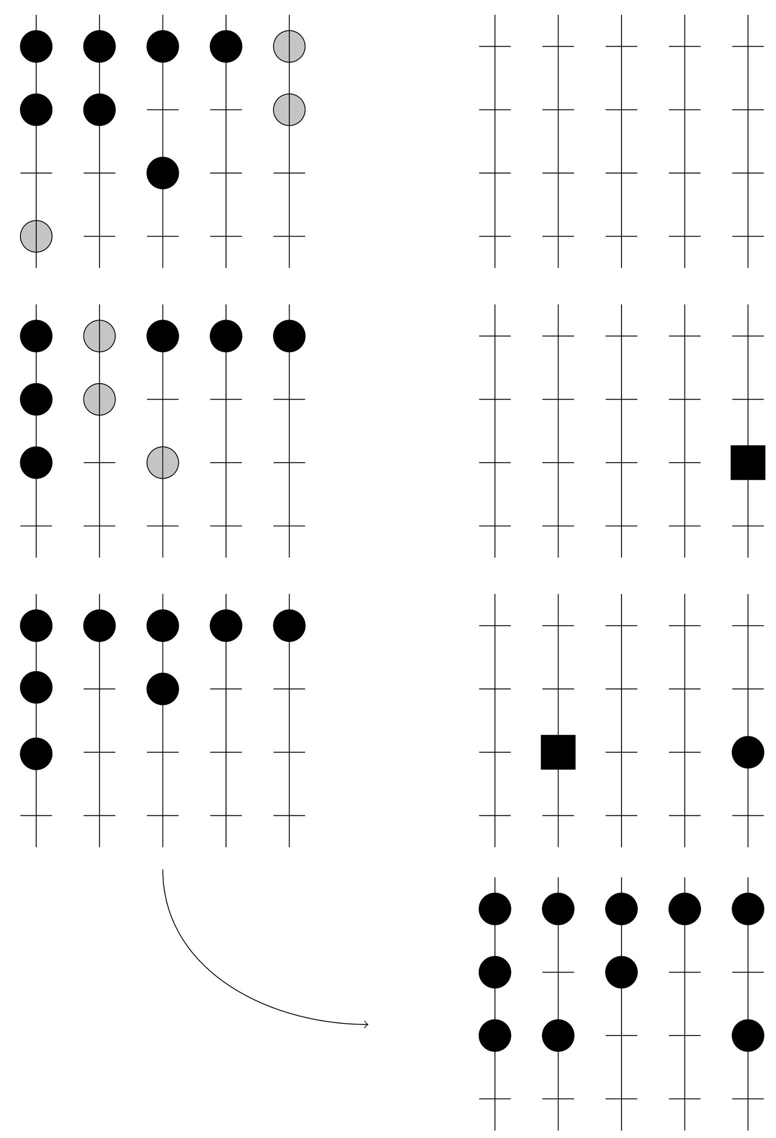

Fig. 6: Starting with two James abaci with $e = 5$ runners, one of $\lambda = (6,4,2)$ and one empty, we repeatedly move left leading beads up and add the new bead on the right. We move all left beads to the right diagram once there are no empty spaces above them. We get $m_5(\lambda)' = (5,3,3,1)$.

Results

Partitions with divisible arms

We say that a partition λ is

- **d-balanced** if the arm lengths of its e-hooks $\equiv 0 \pmod{d}$;
- **d-shift balanced** if the arm lengths of its e-hooks $\equiv -1$ (d);
- **d-skewed** if the arm lengths of its e-hooks $\not\equiv 0$ (d);
- **d-shift skewed** if the arm lengths of its e-hooks $\not\equiv -1$ (d).

The **e-residue** of a box (i,j) of a Young diagram is the reminder of $j - i$ modulo e. The (x,y)-entry of the **d-runner matrix** of λ $(1 \le x \le d-1, 0 \le y \le e-1)$ counts the parts of λ congruent to x modulo d with their rightmost box having e-residue y; see Figure 7.

Theorem (T. 2025+)

Let $1 < d < e$ be coprime integers and λ be an e-regular d-balanced partition. Then $m_e(\lambda)'$ is a d-shift balanced partition with the same d-runner matrix and e-core as λ. Moreover, this determines $m_e(\lambda)'$ uniquely.

Given a partition γ with no e-hooks and a $(d-1) \times e$ matrix $\mathcal{R}$ of non-negative integers, let $E_{\mathcal{R}}(\gamma)$ be the set of partitions of minimal size with e-core γ and d-runner matrix $\mathcal{R}$.

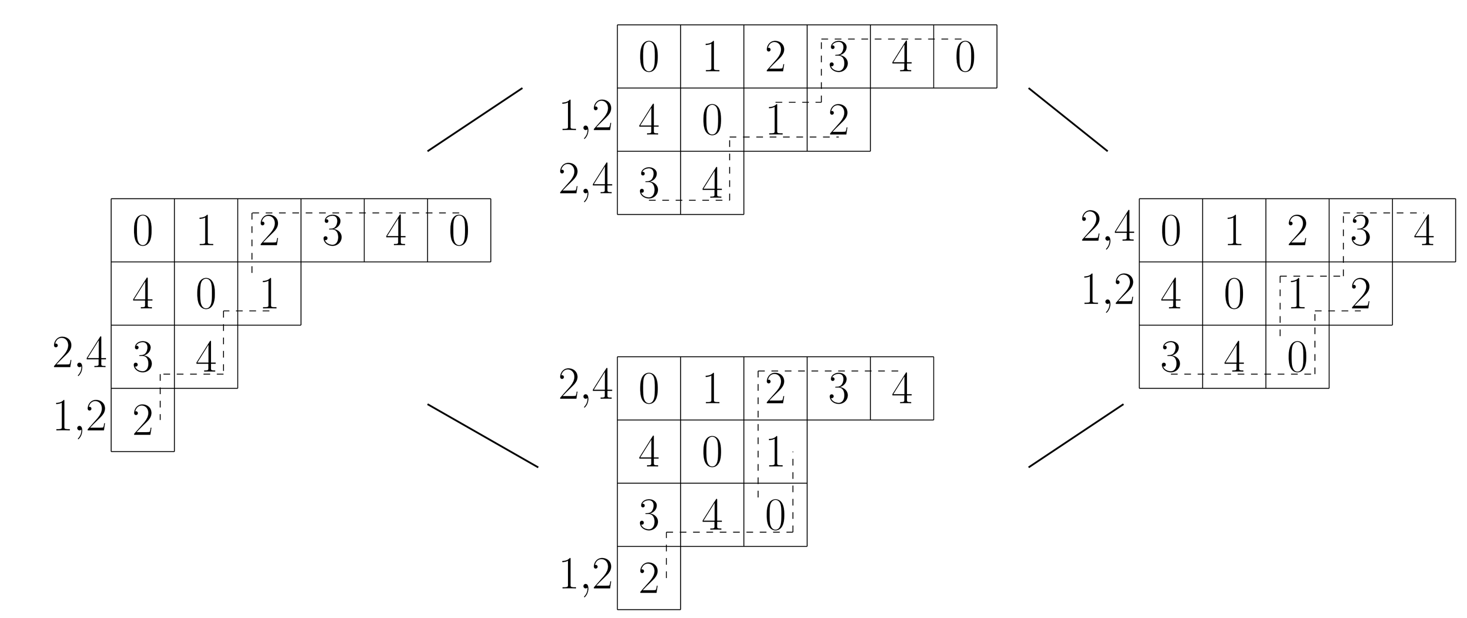

Fig. 7: Members of $E_{\mathcal{R}}(\gamma)$ with $d = 3, \gamma = (1,1)$ and $\mathcal{R} = \begin{pmatrix} 0&0&1&0&0 \\ 0&0&0&0&1 \end{pmatrix}$ with 5-residues and 5-hooks. The top one is 3-balanced and the bottom one is 3-shift balanced; thus $m_5((6,4,2))' = (5,3,3,1)$.

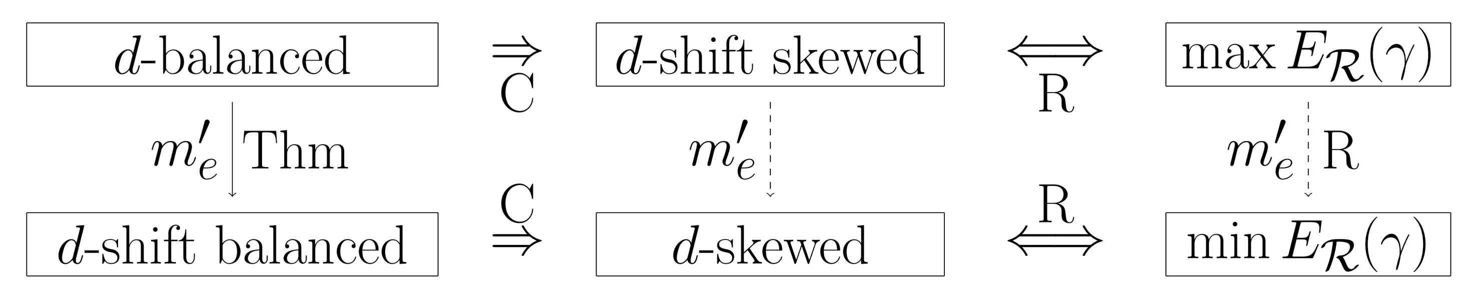

Fig. 8: Summary of further combinatorial results. Here, C = clear and R = new result. When $d = 2$, implications become equivalences and dashed arrows become full arrows.