Mullineux map restricted to certain partitions divisible by d

Pavel Turek

Royal Holloway, University of London

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Outline

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Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ of n is a non-increasing sequence of positive integers, called parts, which add up to n.

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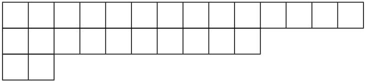

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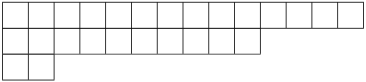

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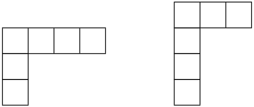
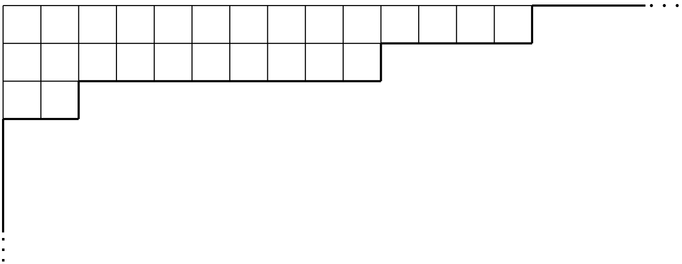


Figure: The conjugate of $(4, 1^2)$ is $(3, 1^3)$.

James abacus

Alternatively, we can represent partitions using James abacus. We form a runner from the border edges of the Young diagram and place beads on the vertical edges.



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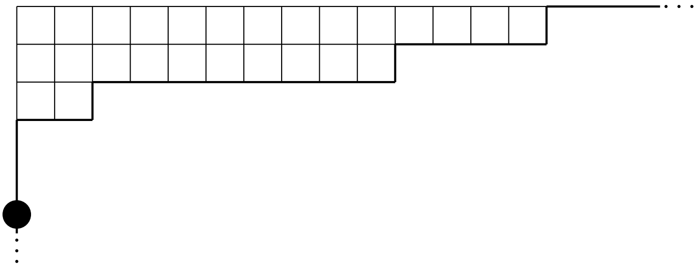

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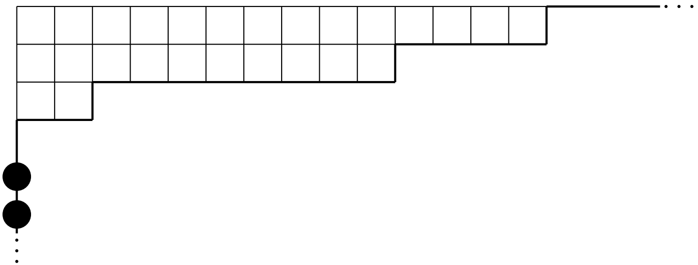

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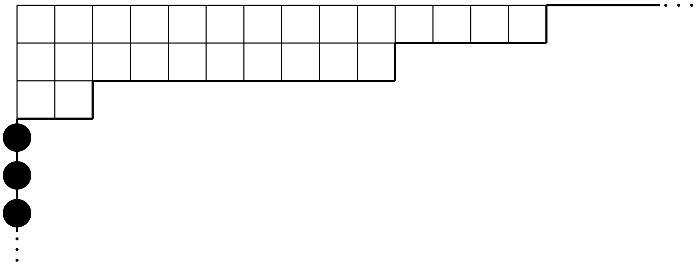

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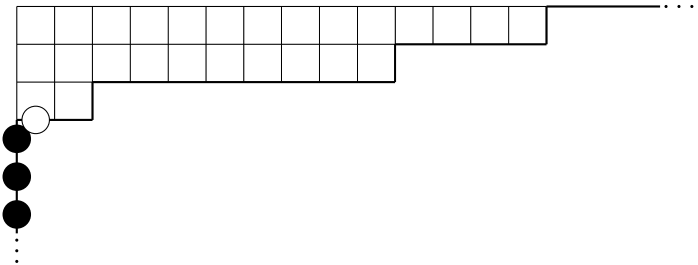

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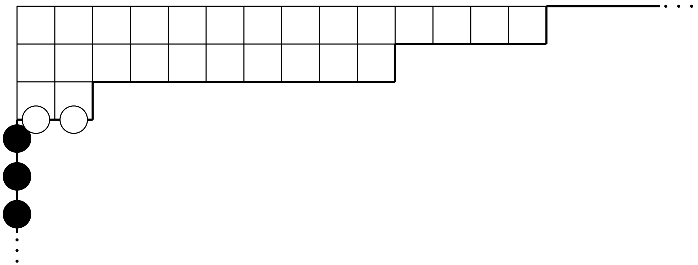

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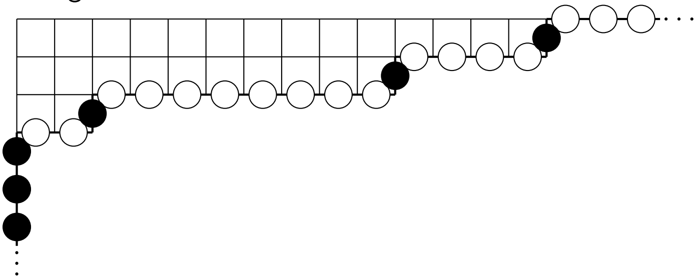

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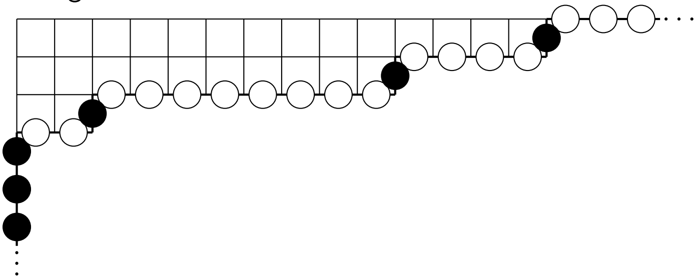

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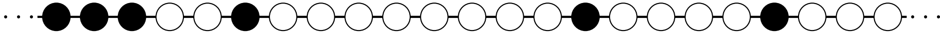

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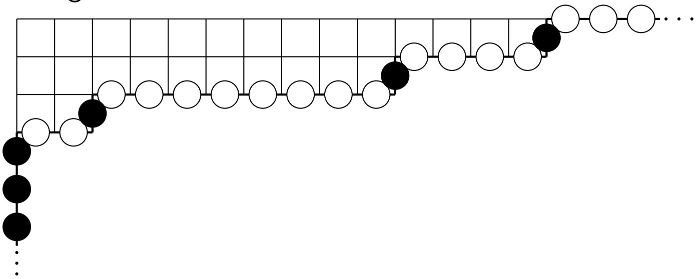

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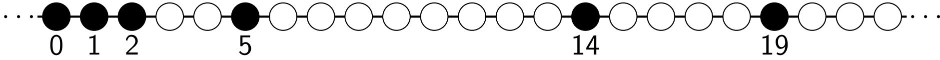

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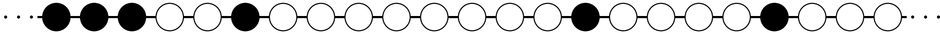
We recover the partition from James abacus by counting unoccupied spaces before each bead.



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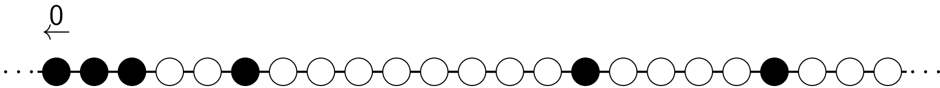

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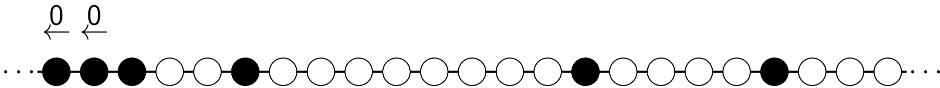

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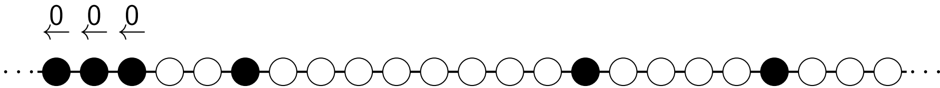

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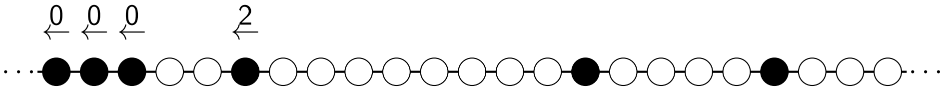

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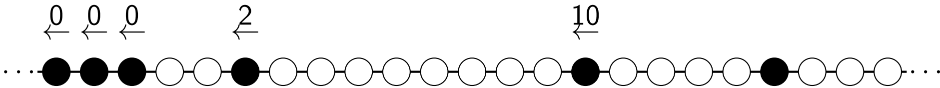

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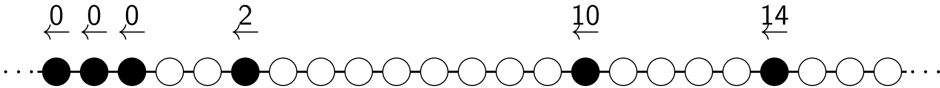

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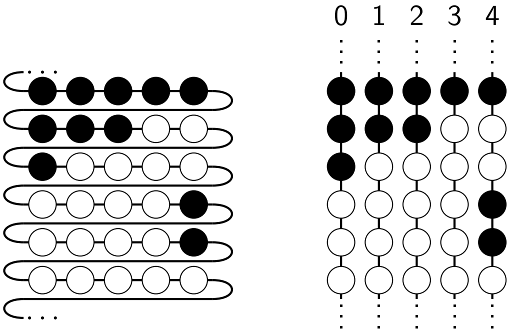

Figure: The 5-abacus of partition $(14, 10, 2)$.

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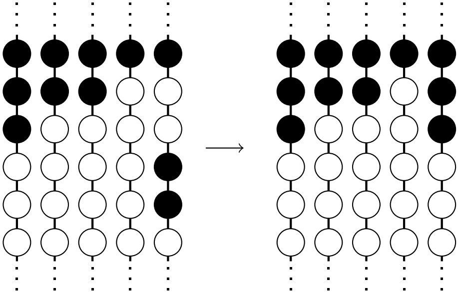

Figure: The 5-core of $(14, 10, 2)$ is $(4, 1^2)$.

e-core

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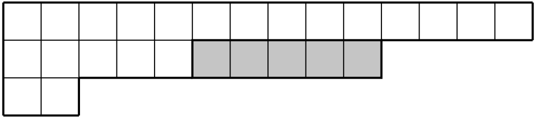

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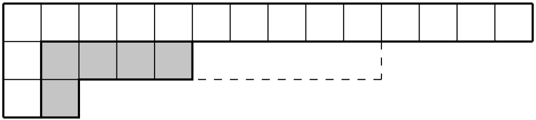

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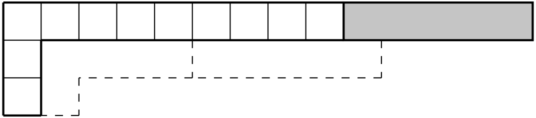

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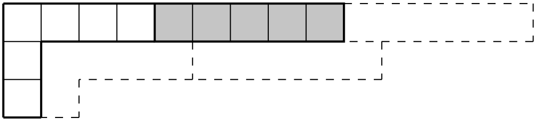

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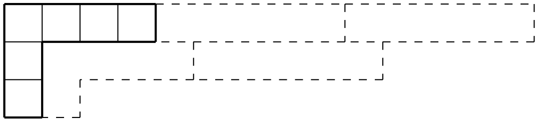

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A partition is *e-regular* if it has no e equal parts.

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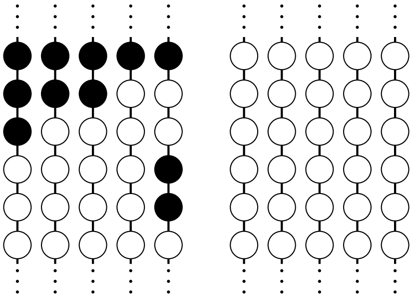
Abacus Mullineux algorithm



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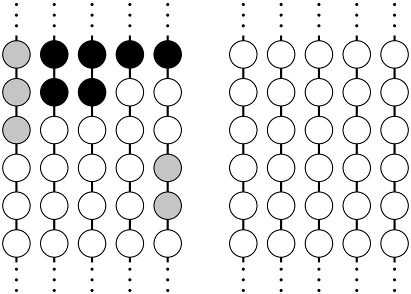

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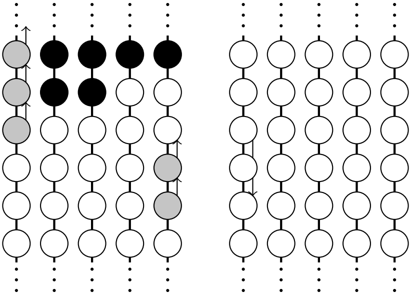

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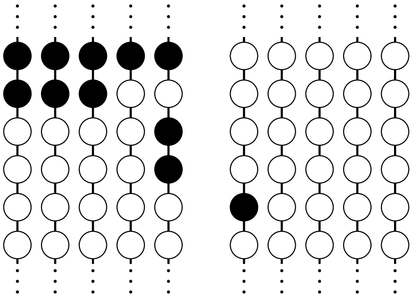

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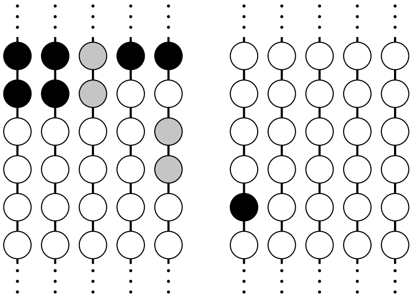

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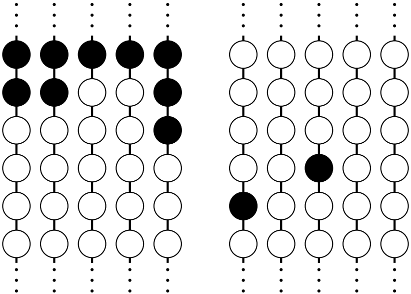

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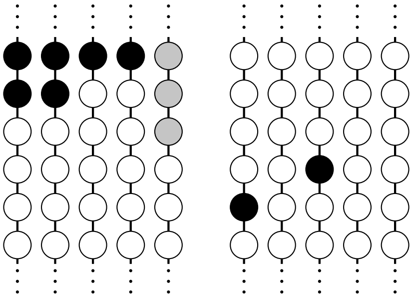

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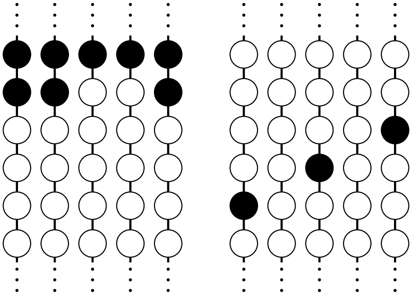

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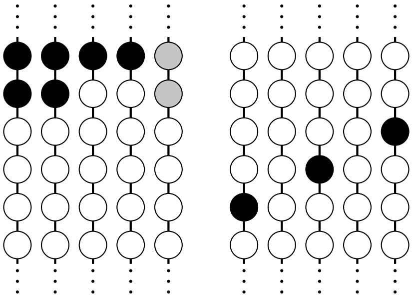

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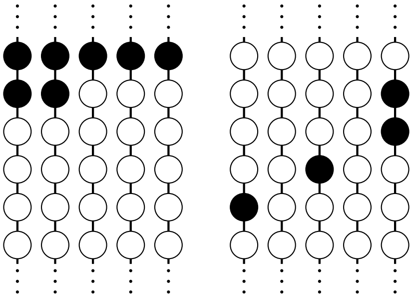

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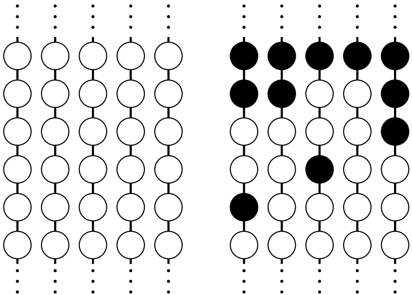

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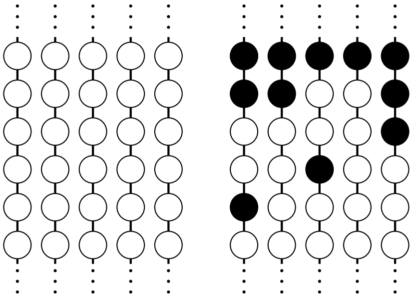

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2. Mullineux map
3. d-maximal and d-minimal partitions
4. Key partitions
5. Finding d-maximal and d-minimal partitions

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Example

Let $e = 5, d = 2$ and $\gamma = (4, 1^2)$.

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- $(14, 10, 2)$,
- $(14, 8, 2, 2)$,
- $(14, 6, 6)$,
- $(14, 6, 4, 2)$,
- $(12, 10, 2, 2)$,
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Thus $(14, 10, 2)$ is 2-maximal and $(10, 8, 6, 2)$ is 2-minimal.

Significance

Giannelli and Wildon in 2015 showed that the set $E^2(\gamma)$ posses information about the decomposition matrix $(d_{\lambda,\mu})$ of symmetric group.

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Our task:

- show that $\widetilde{m_e}$ maps 2-maximal partitions to 2-minimal partitions,
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- describe how to find d-maximal and d-minimal partitions with given e-core.

Outline

d-balanced partitions

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A partition λ is *d-balanced* if

- it is divisible by d, and

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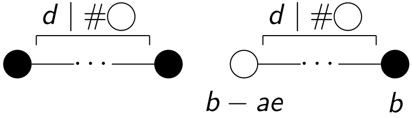

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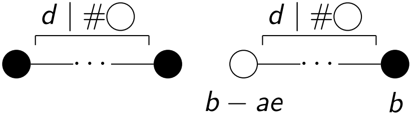

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d-balanced partitions – example

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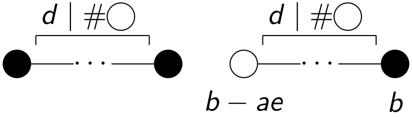

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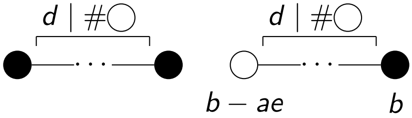

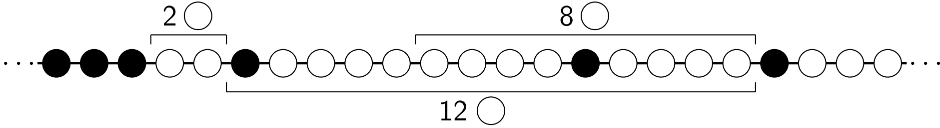

Figure: For $e = 5$, the partition $(14, 10, 2)$ is 2-balanced.

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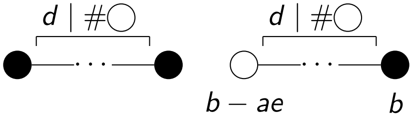

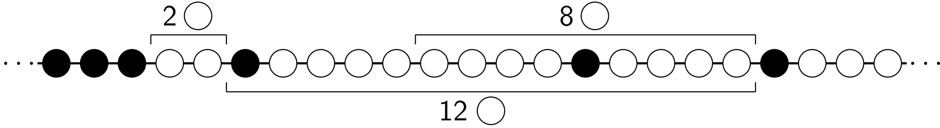

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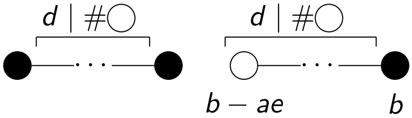

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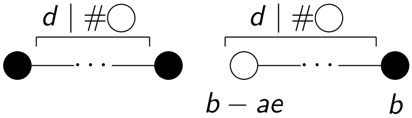

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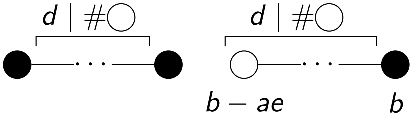

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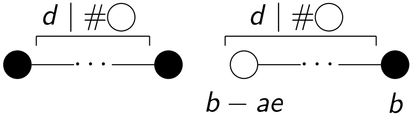

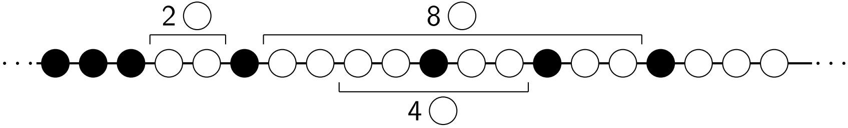

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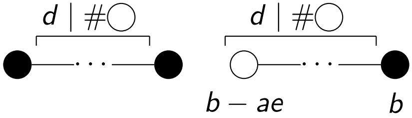

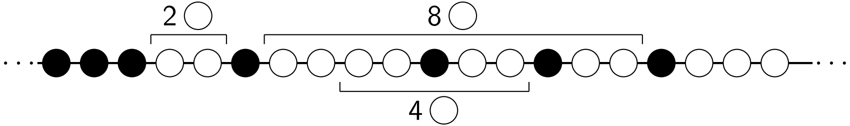

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Theorem A

The relevance of d-balanced partitions and d-shift-balanced partitions to the Mullineux map is as follows.

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The map $\widetilde{m_e}$ sends d-balanced partitions to d-shift-balanced partitions.

This confirms that $(14, 10, 2)$ is sent to $(10, 8, 6, 2)$ by $\widetilde{m_5}$.

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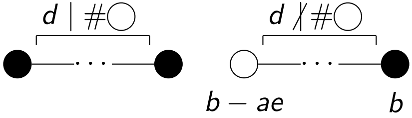

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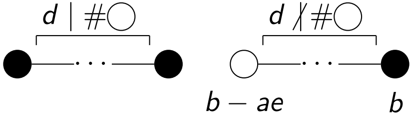

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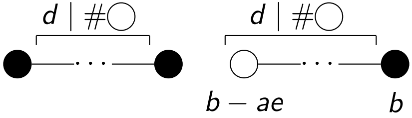

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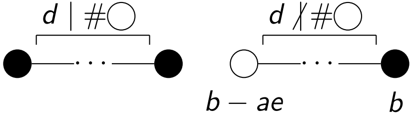

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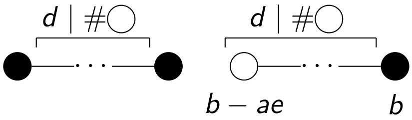

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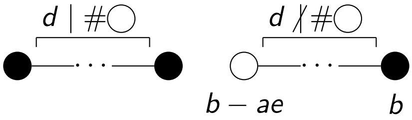

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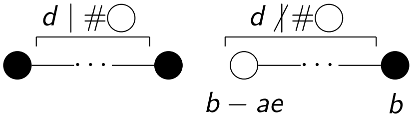

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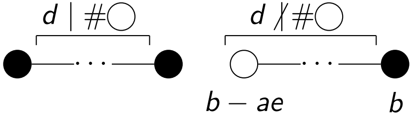

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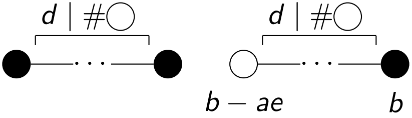

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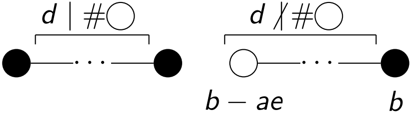

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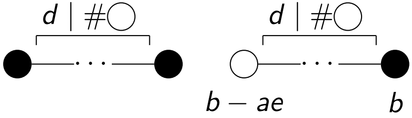

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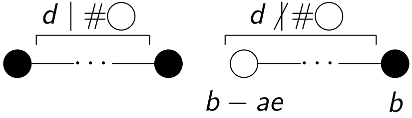

The notions coincide for $d = 2$.

Theorem B and C

The relevance of d-unbalanced and d-shift-unbalanced partitions to our questions comes from the following.

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The d-minimal partitions and d-unbalanced partitions coincide.

Big picture

d-balanced	d-shift-unbalanced	d-maximal
d-shift-balanced	d-unbalanced	d-minimal

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d-balanced $\underset{\text{Clear}}{\Rightarrow}$ d-shift-unbalanced d-maximal

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Theorem (T. 2024+)

Let λ be a d-maximal partition. If λ is d-balanced, then $\widetilde{m_e}(\lambda)$ is the d-minimal partition with the same e-core as λ.

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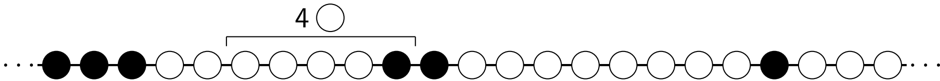
Corollary (T. 2024+)

If λ is 2-maximal, then $\widetilde{m}_e(\lambda)$ is the 2-minimal partition with the same e-core as λ.

Outline

Finding d-maximal partitions

Starting from any partition divisible by d with e-core γ we can find the d-maximal partition with the same e-core by 'removing bad border strips'.



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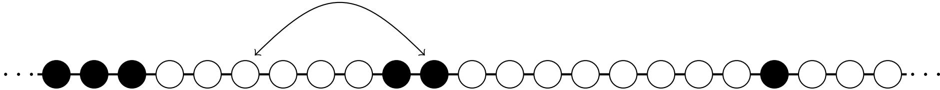

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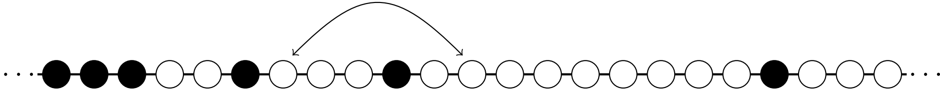

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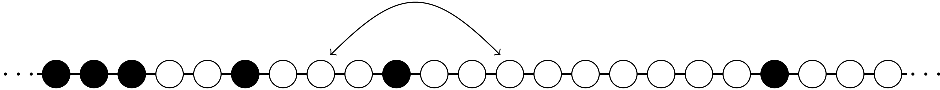

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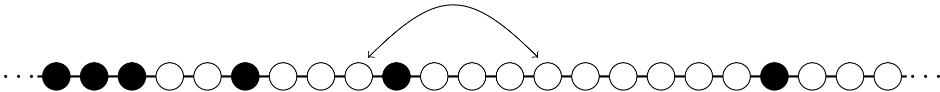
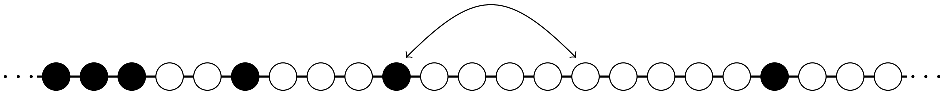


Finding d-maximal partitions

Starting from any partition divisible by d with e-core γ we can find the d-maximal partition with the same e-core by 'removing bad border strips'.



Finding d-maximal partitions

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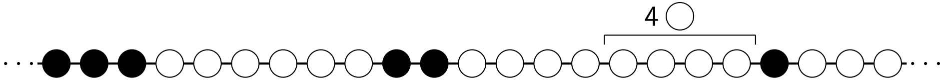
Finding d-maximal partitions

Starting from any partition divisible by d with e-core γ we can find the d-maximal partition with the same e-core by 'removing bad border strips'.

This proves one direction of Theorem B.

Finding d-minimal partitions

We have an analogous algorithm for d-minimal partitions.



Finding d-minimal partitions

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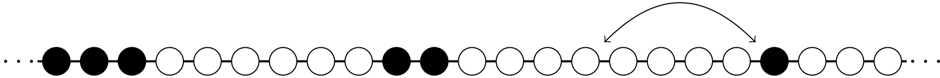

Finding d-minimal partitions

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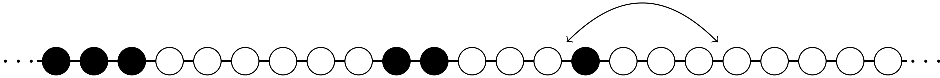

Finding d-minimal partitions

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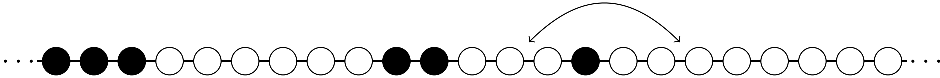

Finding d-minimal partitions

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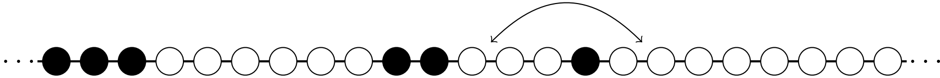

Finding d-minimal partitions

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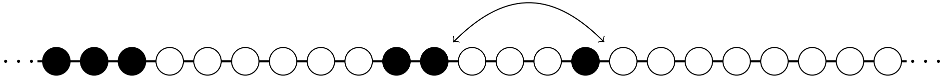
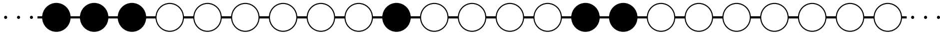


Finding d-minimal partitions

We have an analogous algorithm for d-minimal partitions.



Finding d-minimal partitions

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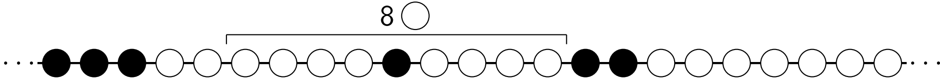

Finding d-minimal partitions

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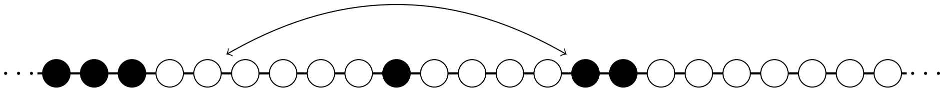
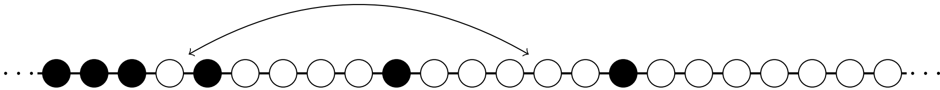


Finding d-minimal partitions

We have an analogous algorithm for d-minimal partitions.



Finding d-minimal partitions

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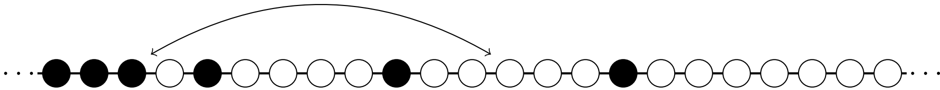

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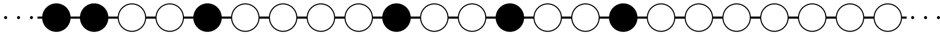

Finding d-minimal partitions

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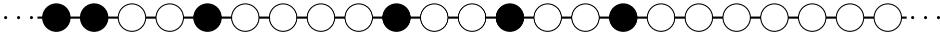

This proves one direction of Theorem C.