The Mullineux map and partitions with arms divisible by d

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Young diagrams

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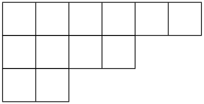

Figure: The Young diagram of $(6, 4, 2)$, a partition of 12.

Conjugate partitions and dominance

The conjugate λ' of λ is a partition which satisfies $Y(\lambda') = \{(j,i) : (i,j) \in Y(\lambda)\}$.

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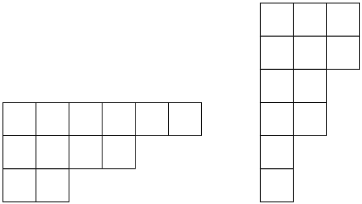

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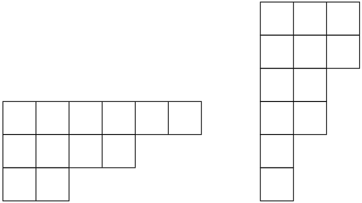

Figure: The conjugate of $(6,4,2)$ is $(3,3,2,2,1,1)$.

For partitions λ and μ of the same size, we say that λ dominates μ if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$$

for all positive integers i (where λ_j, respectively, μ_j is interpreted as 0 if j exceeds the number of parts of the respective partition).

Dominance order – example

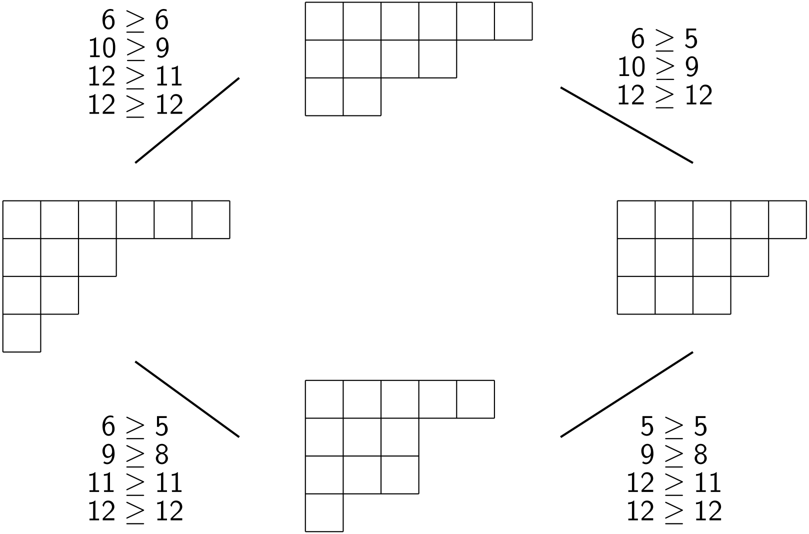

Figure: The Hasse diagram of the four displayed partitions.

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Fix integer $e \geq 2$. Partition λ is e-regular if no e parts of λ are equal.

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The Mullineux map also appears in decomposition matrices.

Mullineux map – decomposition matrix

Let λ be an e-regular partition of n. The unique minimal partition in the dominance order which labels a row of a decomposition matrix with a positive entry in column λ is $m'_e(\lambda) := m_e(\lambda)'$.

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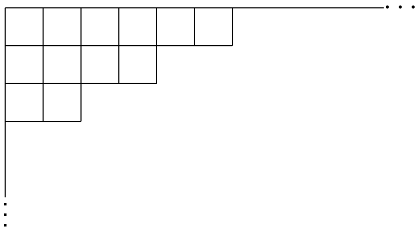
$$
\begin{array}{rcc}
 & & \lambda \\
 & & 0 \\
\lambda & — & 1 \\
 & & 2 \\
 & & 0 \\
m'_e \downarrow & & 1 \\
 & & 4 \\
 & & 1 \\
m_e(\lambda)' & — & 1 \\
 & & 0 \\
 & & 0
\end{array}
$$

Figure: An illustration of column λ of a decomposition matrix.

James abacus

Partitions can be represented on the James abacus: we form a runner of the abacus from the 'inner border' of the Young diagram and place beads on the vertical edges.



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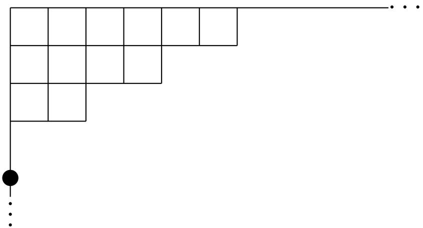

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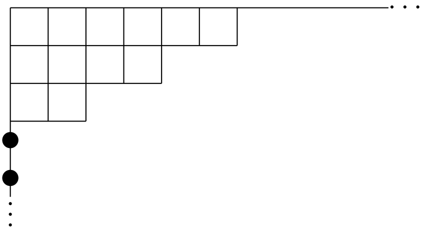

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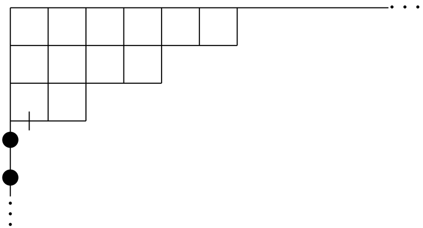

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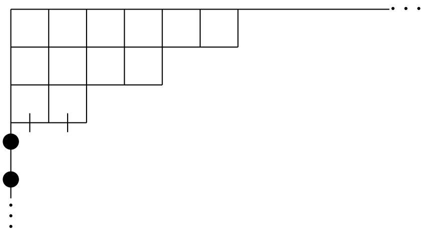

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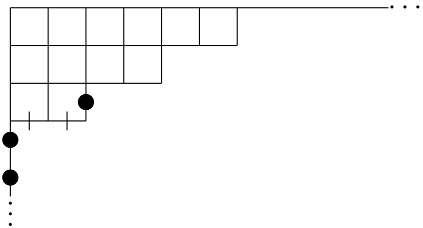

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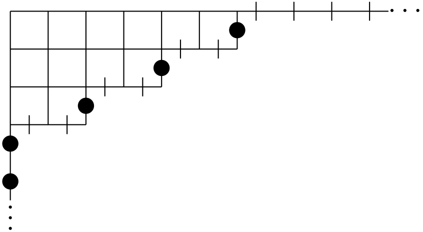

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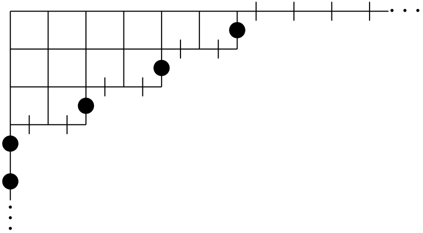

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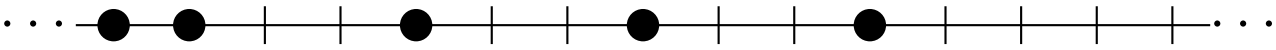

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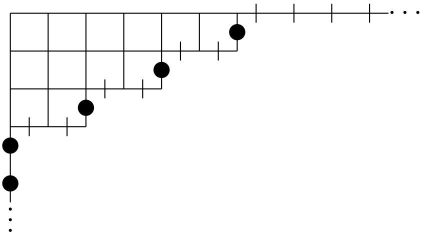

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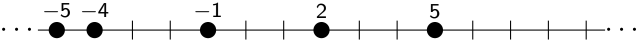

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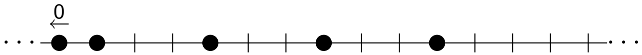

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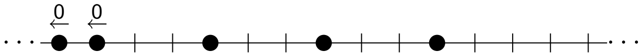

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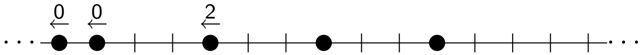

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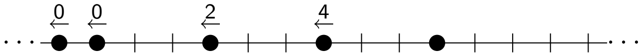

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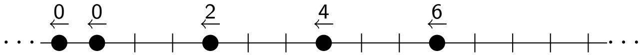

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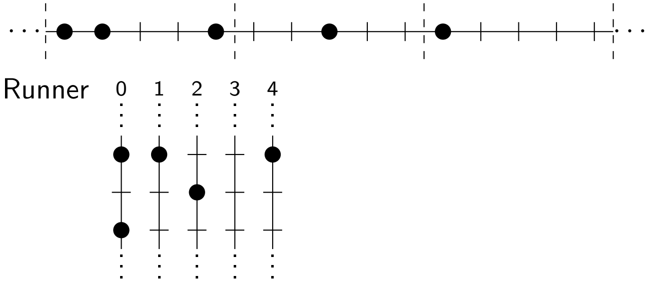

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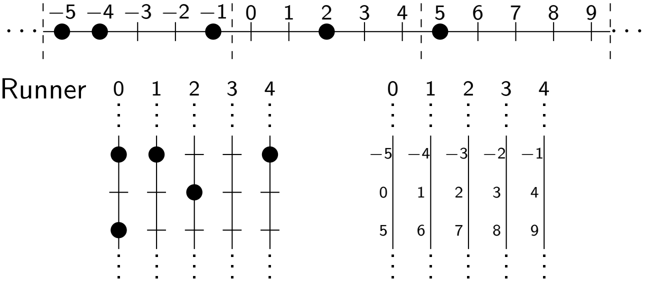

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The e-abacus inherits labelling and order of its positions from the James abacus.

Leading beads

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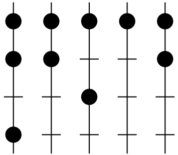

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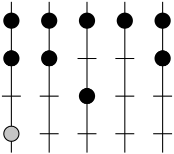

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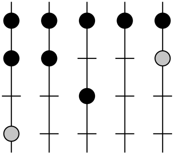

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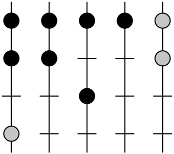

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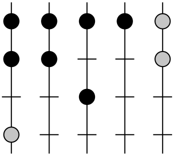

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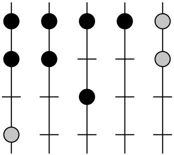

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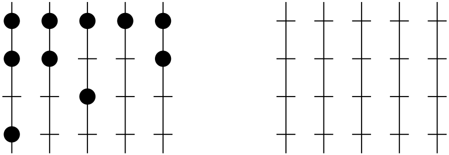
Abacus Mullineux algorithm



Starting with two e-abaci, one of λ and the other empty, we repeatedly identify the leading beads $b_0, b_1, \ldots$ of the first e-abacus, add bead $b_v + ve$ to the second e-abacus, and replace the leading beads by $b_0 - e, b_1 - e, \ldots$

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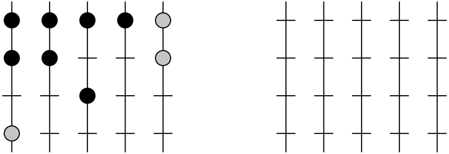

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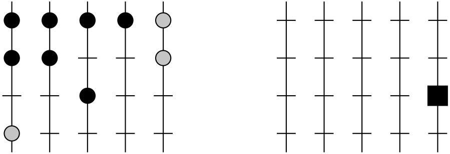

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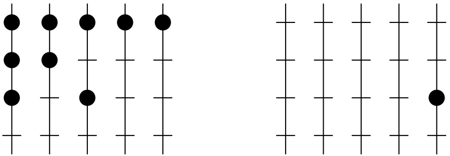

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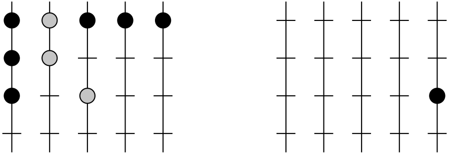

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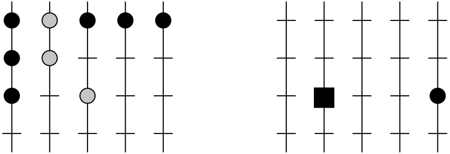

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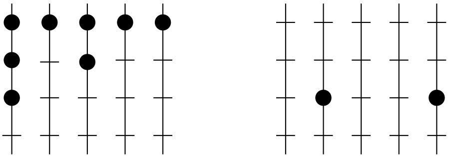

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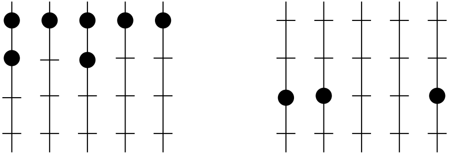

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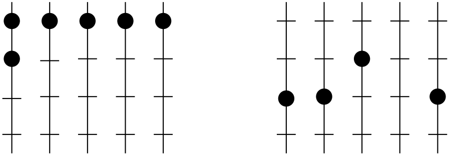

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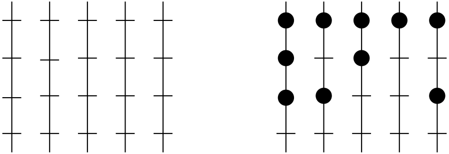

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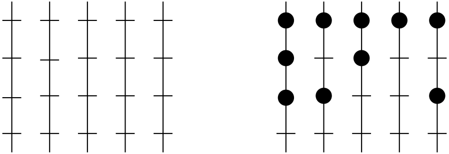
We read $m_e(\lambda)'$ from the other e-abacus.

Abacus Mullineux algorithm



Starting with two e-abaci, one of λ and the other empty, we repeatedly identify the leading beads $b_0, b_1, \ldots$ of the first e-abacus, add bead $b_v + ve$ to the second e-abacus, and replace the leading beads by $b_0 - e, b_1 - e, \ldots$ until we reach the empty partition on the first e-abacus, which we then move to the other one.

We read $m_e(\lambda)'$ from the other e-abacus. Here we get $m_5((6,4,2))' = (5,3,3,1)$.

Big picture of results

Throughout $1 < d < e$ are coprime integers (in examples $e = 5$).

Big picture of results

$$
\begin{tikzcd}
\boxed{d\text{-balanced}} \arrow[r, phantom, "\underset{\text{Clear}}{\Rightarrow}"] \arrow[d, "m'_e"', "\text{Thm A}"] & \boxed{d\text{-shift skewed}} \arrow[r, phantom, "\underset{\text{Prop B}}{\Longleftrightarrow}"] \arrow[d, dashed, "m'_e"'] & \boxed{\max E_{\mathcal{R}}(\gamma)} \arrow[d, dashed, "m'_e"', "\text{Thm D}"] \\
\boxed{d\text{-shift balanced}} \arrow[r, phantom, "\underset{\text{Clear}}{\Rightarrow}"] & \boxed{d\text{-skewed}} \arrow[r, phantom, "\underset{\text{Prop C}}{\Longleftrightarrow}"] & \boxed{\min E_{\mathcal{R}}(\gamma)}
\end{tikzcd}
$$

Big picture of results

$$\begin{tikzcd}
\boxed{\text{2-balanced}} \arrow[r, phantom, "\overset{\Longleftrightarrow}{\text{Clear}}"] \arrow[d, "m'_e"', "\text{Thm A}"] & \boxed{\text{2-shift skewed}} \arrow[r, phantom, "\overset{\Longleftrightarrow}{\text{Prop B}}"] \arrow[d, "m'_e"'] & \boxed{\max E_{\mathcal{R}}(\gamma)} \arrow[d, "m'_e"', "\text{Thm D}"] \\
\boxed{\text{2-shift balanced}} \arrow[r, phantom, "\overset{\Longleftrightarrow}{\text{Clear}}"] & \boxed{\text{2-skewed}} \arrow[r, phantom, "\overset{\Longleftrightarrow}{\text{Prop C}}"] & \boxed{\min E_{\mathcal{R}}(\gamma)}
\end{tikzcd}$$

The results become stronger when $d = 2$.

e-core

A connected removable group of ae boxes (for some $a > 0$) with no 2×2 square is called an e-divisible hook. Its arm-length equals the number of columns the e-divisible hook spans minus one.

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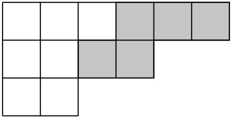

Figure: A 5-divisible hook of $(6, 4, 2)$ of arm length 3.

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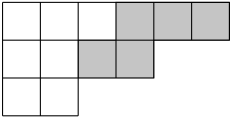

Figure: A 5-divisible hook of $(6, 4, 2)$ of arm length 3.

The e-core of a partition is obtained by successive removals of its e-divisible hooks.

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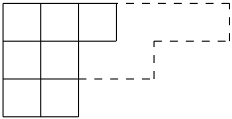

Figure: Finding the 5-core of $(6, 4, 2)$.

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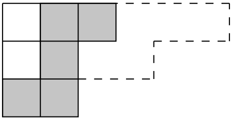

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The e-core of a partition is obtained by successive removals of its e-divisible hooks.

A connected removable group of ae boxes (for some $a > 0$) with no 2×2 square is called an *e*-divisible hook. Its arm-length equals the number of columns the *e*-divisible hook spans minus one.

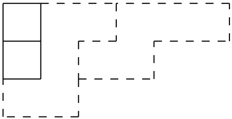

Figure: Finding the 5-core of $(6, 4, 2)$.

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Balanced partitions

Definition

We say that a partition λ is

- d-balanced if the arm lengths of its e-divisible hooks $\equiv 0 \bmod d$;

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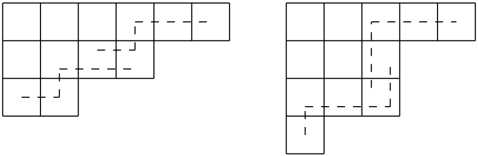

Figure: The first partition is 3-balanced (and 3-shift skewed).

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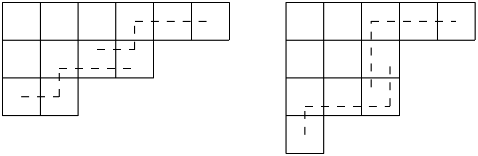

Figure: The first partition is 3-balanced (and 3-shift skewed) and 2-shift balanced (and 2-skewed).

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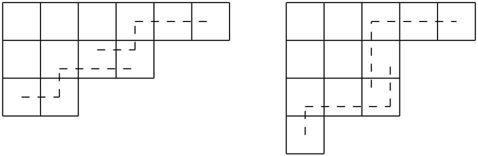

Figure: The first partition is 3-balanced (and 3-shift skewed) and 2-shift balanced (and 2-skewed). Swap 2 and 3 to get statements for the second partition.

d-runner matrix

The e-residue of a box (i, j) of a Young diagram is the reminder of $j - i$ modulo e.

d-runner matrix

The *e-residue* of a box (i, j) of a Young diagram is the reminder of $j - i$ modulo e.

0	1	2	3	4	0
4	0	1	2		
3	4				

0	1	2	3	4
4	0	1		
3	4	0		
2				

Figure: The e-residues of partitions $(6, 4, 2)$ and $(5, 3, 3, 1)$.

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The *d-runner matrix* of λ stores in row x $(1 \leq x \leq d - 1)$ and column y $(0 \leq y \leq e - 1)$ the number of parts of λ which have reminder x modulo d and their rightmost box has e-residue y.

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The *d-runner matrix* of λ stores in row x $(1 \leq x \leq d - 1)$ and column y $(0 \leq y \leq e - 1)$ the number of parts of λ which have reminder x modulo d and their rightmost box has e-residue y. The partitions above have 3-runner matrix $\left(\begin{smallmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right)$.

Theorem A

Theorem (T. 2025+)

Let $1 < d < e$ be coprime integers and λ be an e-regular d-balanced partition. Then $m_e(\lambda)'$ is a d-shift balanced partition with the same d-runner matrix and e-core as λ. Moreover, this determines $m_e(\lambda)'$ uniquely.

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Partitions $(6,4,2)$ and $(5,3,3,1)$ have e-core $(1,1)$, 3-runner matrix $\left(\begin{smallmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right)$, the first one is 3-balanced, and the second one is 3-shift balanced.

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Partitions $(6, 4, 2)$ and $(5, 3, 3, 1)$ have e-core $(1, 1)$, 3-runner matrix $\left(\begin{smallmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right)$, the first one is 3-balanced, and the second one is 3-shift balanced.

Hence $m_e((6, 4, 2))' = (5, 3, 3, 1)$.

Set $E_{\mathcal{R}}(\gamma)$

Given a partition γ with no e-divisible hooks and a $(d-1)\times e$ matrix $\mathcal{R}$ of non-negative integers, let $E_{\mathcal{R}}(\gamma)$ be the set of partitions of minimal size with e-core γ and d-runner matrix $\mathcal{R}$.

Set $E_{\mathcal{R}}(\gamma)$

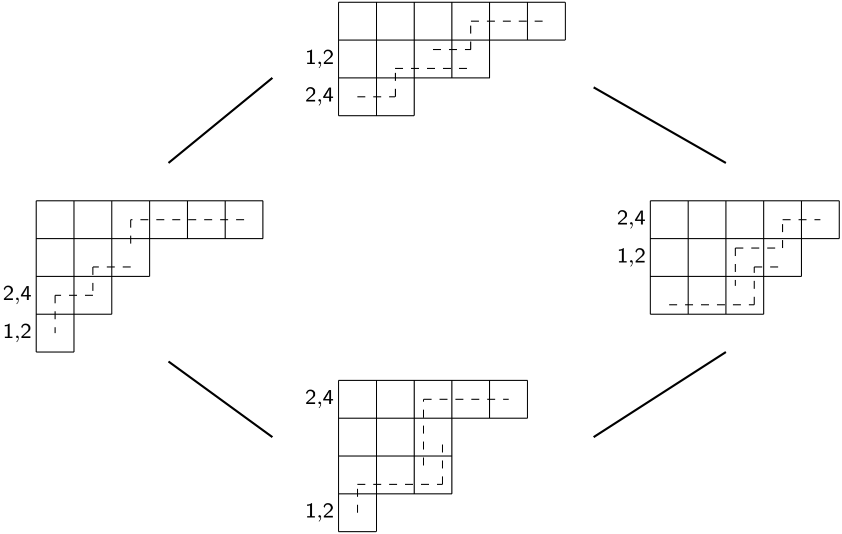

Figure: Partitions in the set $E_{\mathcal{R}}(\gamma)$ with $\gamma = (1,1)$ and $\mathcal{R} = \left(\begin{smallmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right)$.

Propositions B and C

Proposition (T. 2025+)

Let $1 < d < e$ be coprime integers and λ be a partition with e-core γ and d-runner matrix $\mathcal{R}$. Then λ is d-shift skewed if and only if it is the maximal element of $E_{\mathcal{R}}(\gamma)$.

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Proposition (T. 2025+)

Let $1 < d < e$ be coprime integers and λ be a partition with e-core γ and d-runner matrix $\mathcal{R}$ such that each row of $\mathcal{R}$ contains 0. Then λ is d-skewed if and only if it is the minimal element of $E_{\mathcal{R}}(\gamma)$.

Theorem D

Theorem (T. 2025+)

Let $1 < d < e$ be coprime integers and γ be an e-core partition and $\mathcal{R}$ a $(d-1) \times e$ matrix of non-negative integers with at least one 0 in its first row. The set $E_{\mathcal{R}}(\gamma)$ has a unique maximal element λ in the dominance order. Moreover, if λ is d-balanced, then $m_e(\lambda)'$ is a unique minimal partition of $E_{\mathcal{R}}(\gamma)$ in the dominance order.

Big picture of results

$$
\begin{tikzcd}
\boxed{d\text{-balanced}} \arrow[r, Rightarrow, "\text{Clear}"'] \arrow[d, "m'_e"', "\text{Thm A}"] & \boxed{d\text{-shift skewed}} \arrow[r, Leftrightarrow, "\text{Prop B}"'] \arrow[d, dashed, "m'_e"'] & \boxed{\max E_{\mathcal{R}}(\gamma)} \arrow[d, dashed, "m'_e"', "\text{Thm D}"] \\
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\end{tikzcd}
$$

Big picture of results

$$
\begin{tikzcd}
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$$