MULLINEUX MAP RESTRICTED TO CERTAIN EVEN PARTITIONS

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Abstract. Let p be a prime. The Mullineux map m_p maps a p-regular partition λ to a p-regular partition μ such that over a field of characteristic p the irreducible modules of a symmetric group $D^\lambda \otimes \mathrm{sgn}$ and D^μ are isomorphic. In 1979, Mullineux conjectured a combinatorial description of the map m_p, which has been since, together with many variations, proved. In the talk we use the combinatorial description by Xu put on an abacus to deduce an interesting behaviour of the Mullineux map on a set of certain 'maximal' even partitions.

1. Partitions

Partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)$ of n is a non-increasing sequence of positive integers, called *parts*, which add up to n. We represent partitions using *Young diagrams* $Y(\lambda) = \left\{(i,j) \in \mathbb{N}^2 : i \leq t, j \leq \lambda_i\right\}$. The *conjugate* λ' of λ satisfies $Y(\lambda') = \{(j,i) : (i,j) \in Y(\lambda)\}$. See Figure 1 for an example.

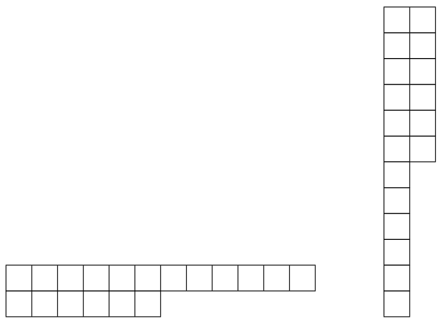

Figure 1. The Young diagrams of $(12,6)$, a partition of 18, and its conjugate $(2^6, 1^6)$.

Alternative representation uses *James abacus* and is obtained as follows. We extend the Young diagram by an infinite vertical line before the first column and an infinite horizontal line above the first row as done on Figure 2. We now attach a bead to the rightmost vertical edge in each row and an empty position to the bottommost horizontal edge in each column. We then form a runner from these edges; see Figure 3 for an example. Note that there is a clear total order on the positions and the process can be reversed by counting the empty positions before each bead.

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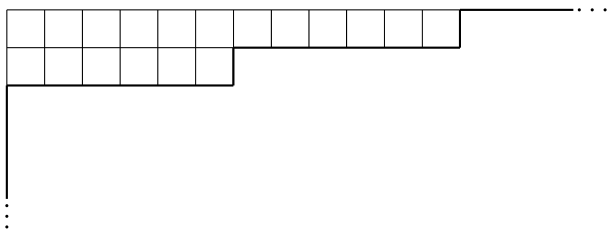

FIGURE 2. The first step in creating the James abacus of $(12, 6)$.

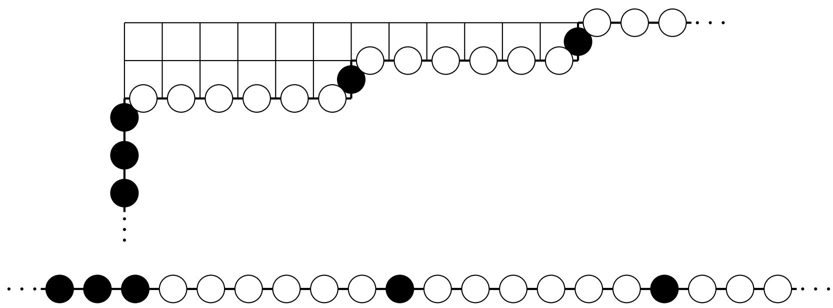

FIGURE 3. The James abacus of $(12, 6)$.

We often replace James abacus by a p-*abacus* which has p runners and is obtained by forming row a from positions $ap, ap + 1, \ldots, ap + p - 1$. We keep the total order from the James abacus. See Figure 4 for an illustrative example.

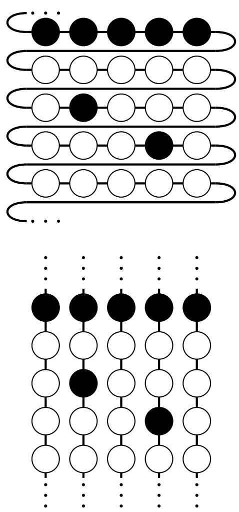

FIGURE 4. The first diagram is the James abacus $(12, 6)$ with runner folded, so each row contains $p = 5$ positions. The second diagram is then the 5-abacus of $(12, 6)$.

The final term to define is p-core. The *p-core* of a partition λ is obtained by sliding all beads on the p-abacus of λ upwards. An example is shown in Figure 5.

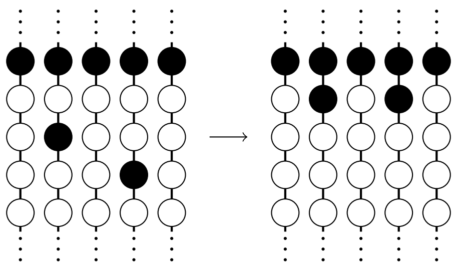

FIGURE 5. The 5-core of $(12, 6)$ is $(2, 1)$.

One can alternatively, obtain the p-core of λ by removing *p-border strips* (p edge-connected boxes of the Young diagram with no 2×2 square, removal of which yields a Young diagram of a partition) from λ until there is none left to be removed. This is well-defined. See Figure 6 for an example.

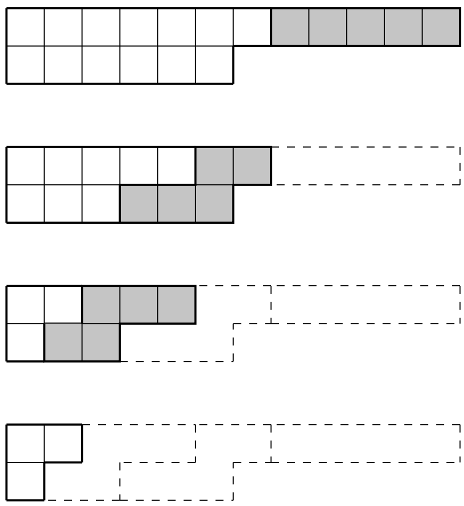

FIGURE 6. The removal of 3 5-border strips of $(12, 6)$ which results in its 5-core $(2, 1)$.

2. MULLINEUX MAP

Fix prime p. We say that partition λ is *p-regular* if there are no equal p parts. We need the following result from the representation theory of symmetric groups.

Theorem 2.1 (James 1976)**.** *The irreducible modules of symmetric groups S_n in characteristic p are D^λ labelled by p-regular partitions of n.*

Remark 2.2. This does not say much as we have not defined the modules anywhere. What we should take from this result is the set of labels.

Given a p-regular partition λ, the module $D^\lambda \otimes \mathrm{sgn}$ is irreducible and so there is a p-regular partition μ such that $D^\lambda \otimes \mathrm{sgn} \cong D^\mu$. We define the *Mullineux map* m_p by $m_p(\lambda) = \mu$ and we write $\widetilde{m_p}$ for the Mullineux map followed by conjugation.

Remark 2.3. Using Hecke algebra and its involution, one can define algebraically m_e for any integer $e > 1$. The results presented here should work (sometimes under mild changes) for any $e > 1$.

There are many combinatorial/algorithmic descriptions of the Mullineux map. For instance, there is a description by Mullineux [Mul79] (proved by Ford and Kleshchev [FK97]), by Xu [Xu97], by Brundan, Kujawa [BK03], by Jacon [Jac21], by Fayers [Fay22] and others. While most of these algorithms are easy to understand it still require effort to prove new properties of the Mullineux map.

An easy property to see is that $\widetilde{m_p}$ preserves p-cores.

Using the algorithm of Xu, we obtain the following 'abacus Mullineux algorithm': We start with two p-abaci, the original one, called A, and the recovery one, called B. Initially, A is the p-abacus of λ and B is the p-abacus. In each step we identify the leading beads $b_0, b_1, \dots$ such that the position of b_i is the largest occupied position of A which is at most the position above b_{i-1}.

There is an index i such that $b_i, b_{i+1}, \dots$ lie on the same runner next to each other. We add a bead to B to the position of b_i shifted by i places downwards. We then move leading beads on A by 1 place upwards. Once A is empty, we let $am_p(\lambda)$ be the partition of B.

Since the algorithm is technically infinite, formally, we stop at the moment when A is a p-core. We then copy A onto B. See Figure 7 and Figure 8 for the algorithm.

Proposition 2.4. *If λ is a p-regular partition, then $am_p(\lambda) = \widetilde{m_p}(\lambda)$.*

3. Main result

The main result describes the Mullineux map for particular partitions. There are other results of this type, for instance the description of the Mullineux map for RoCK blocks [Pag06]. We need some more definitions. We fix an integer $1 < d < p$.

Definition 3.1. Let μ be a p-core partition and let $E_d(\mu)$ be the set of partitions divisible by d with p-core μ which can be obtained from μ by adding the least possible number of p-border strips. Write $\rho(\mu)$ and $\tau(\mu)$ for the lexicographical maximal and minimal element of $E_d(\mu)$, respectively.

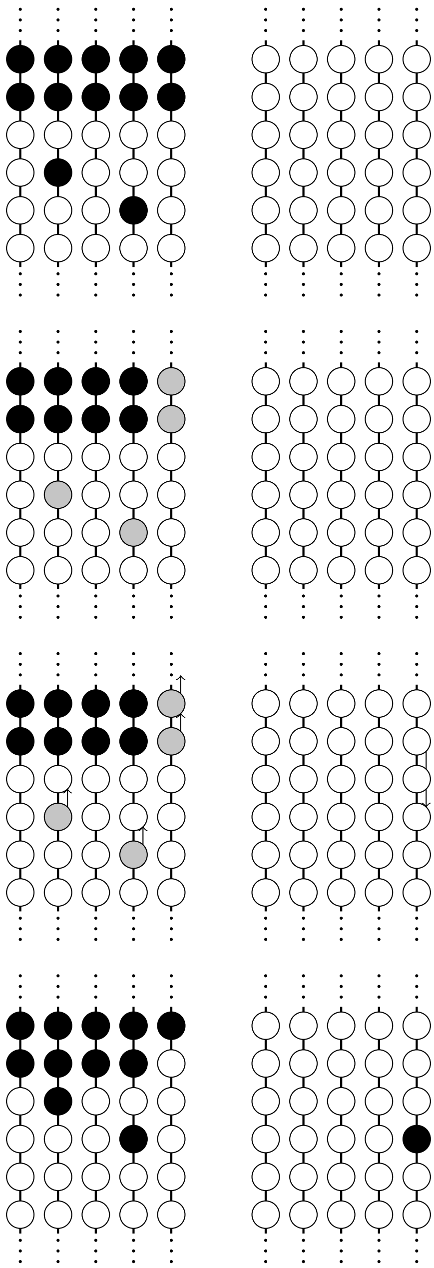

FIGURE 7. The first part of the 'abacus Mullineux algorithm' applied to partition $(12, 6)$.

Partitions which arise as $\rho(\mu)$ and $\tau(\mu)$ (for some μ) are called (p, d)-*maximal* and (p, d)-*minimal*, respectively.

Example 3.2. Let $p = 5, d = 2$ and $\mu = (2, 1)$. Since μ is a partition of an odd integer, namely 3, we need to add odd number of 5-border strips to get an even partition. One checks that one is not sufficient and $E_d(\mu) = \{(12, 6), (10, 8), (8, 8, 2), (8, 6, 4)\}$. Thus $\rho(\mu) = (12, 6)$ and $\tau(\mu) = (8, 6, 4)$.

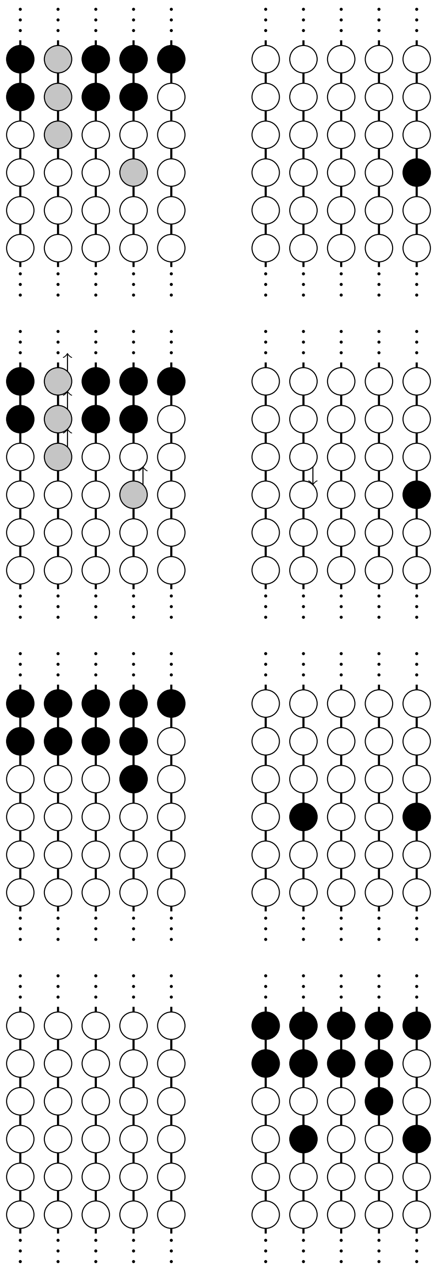

Figure 8. The second part of the 'abacus Mullineux algorithm' applied to partition $(12, 6)$. It results in partition $(8, 6, 4)$.

Theorem 3.3 (T. 2024+)**.** *If λ is $(p, 2)$-maximal, then $\widetilde{m_p}(\lambda)$ is the $(p, 2)$-minimal partition with the same p-core as λ.*

Example 3.4. $\widetilde{m_p}((12, 6)) = (8, 6, 4)$.

The proof consist of two parts which utilise the following definition.

Definition 3.5. A partition λ is (p, d)-*balanced* if it is divisible by d and for any $a > 0$ and any ap consecutive positions on the James abacus of λ starting with a bead (on the right), the number of the empty position on these positions is divisible by d. If the second condition holds for any $ap+1$ positions (rather than ap), then λ is (p, d)-*shift-balanced*.

Note that thanks to the condition that λ is divisible by d we only need to check ap (respectively $ap + 1$) positions such that the next position on the left (the last position on the left) is empty. See Figure 9 and Figure 10 for examples.

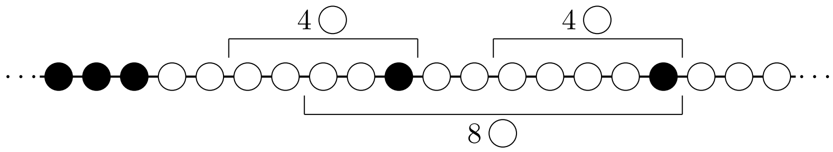

FIGURE 9. The partition $(12, 6)$ is $(5, 2)$-balanced.

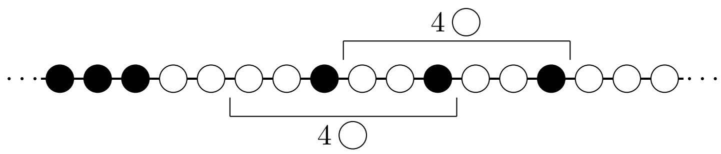

FIGURE 10. The partition $(8, 6, 4)$ is $(5, 2)$-shift-balanced.

The two results we need are as follows.

Theorem 3.6 (T. 2024)**.** *The map $\widetilde{m_p}$ sends (p, d)-balanced partitions to (p, d)-shift-balanced partitions.*

Proposition 3.7. *The $(p, 2)$–maximal (respectively $(p, 2)$-minimal) partitions are precisely the $(p, 2)$-balanced ($(p, 2)$-shift balanced) partitions.*

4. Open questions

We would like to have an analogue to the the previous result for any d. We need an extra piece of notation.

Definition 4.1. A partition λ is (p, d)-*unbalanced* if it is divisible by d and for any $a > 0$ and any ap consecutive positions on the James abacus of λ starting with a bead (on the right) and *followed by an empty position* on the left, the number of the empty position there is *not* divisible by d. If the second condition holds for any $ap + 1$ positions (rather than ap) *ending in an empty position*, then λ is (p, d)-*shift-unbalanced*.

Proposition 4.2. *The (p, d)-maximal partitions and (p, d)-shift-unbalanced partitions coincide.*

Conjecture 4.3. *The (p, d)-minimal partitions and (p, d)-unbalanced partitions coincide.*

Consequently, any (p, d)-balanced partition is (p, d)-maximal. However, the converse is not true. One would like to know if there is any reasonable restriction imposed on the p-cores which would generalise the previous results to $d > 2$.

References

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