Mullineux map restricted to certain even partitions

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Outline

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Young diagrams

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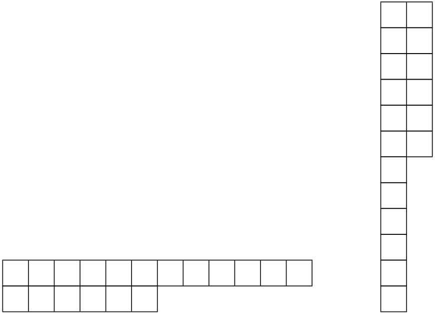

Figure: The Young diagrams of $(12, 6)$, a partition of 18, and its conjugate $(2^6, 1^6)$.

James abacus

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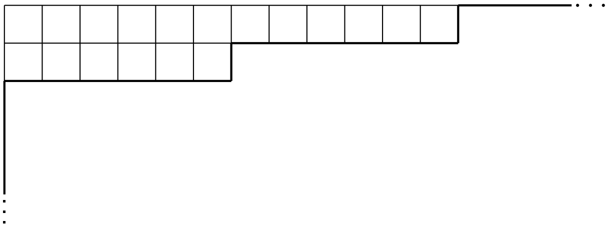

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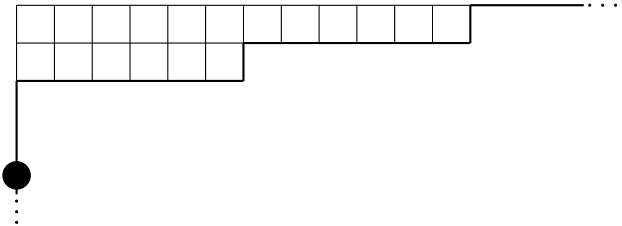

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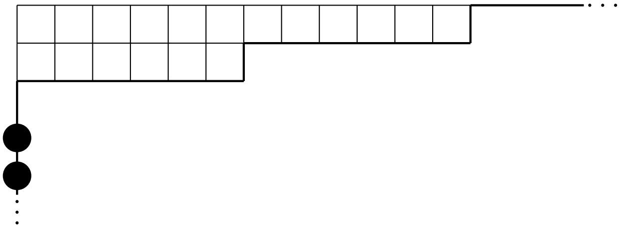

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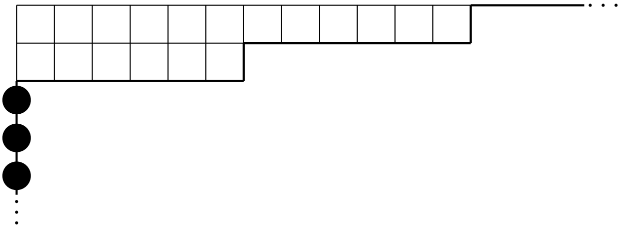

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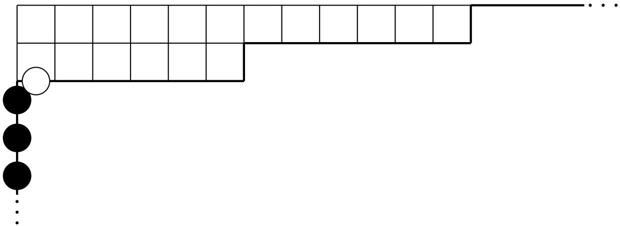

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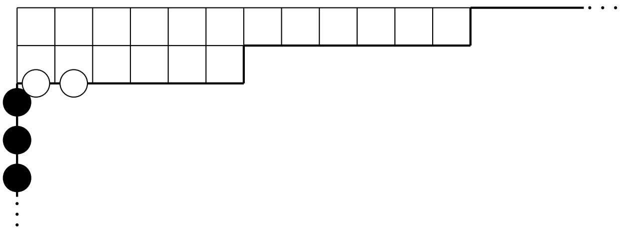

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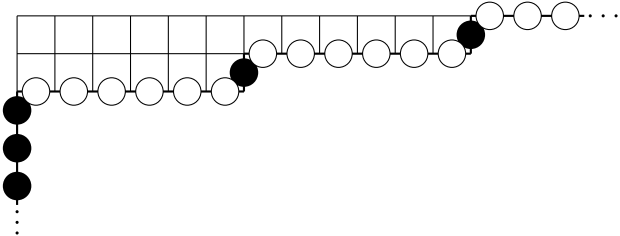

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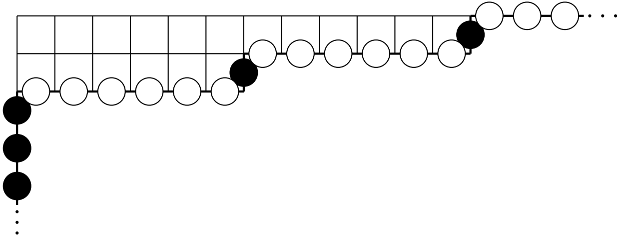

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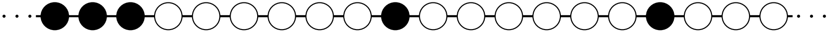

Figure: The James abacus of $(12, 6)$.

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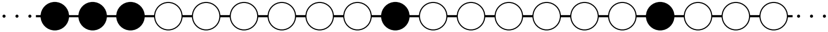

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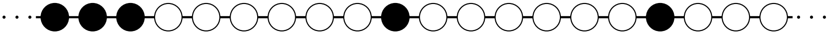

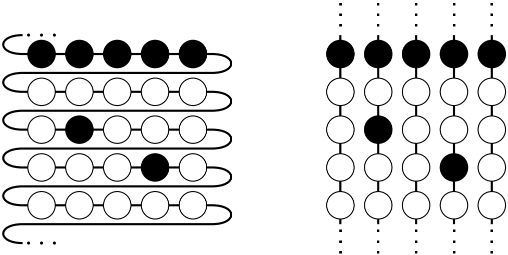

Figure: The 5-abacus of partition $(12, 6)$.

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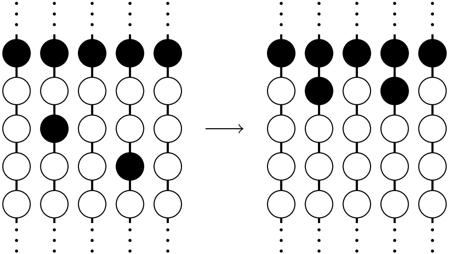

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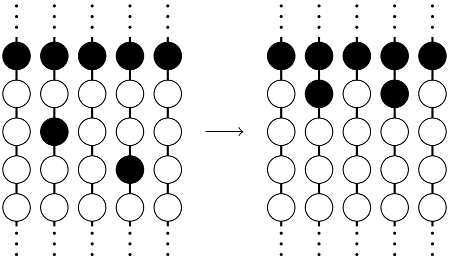

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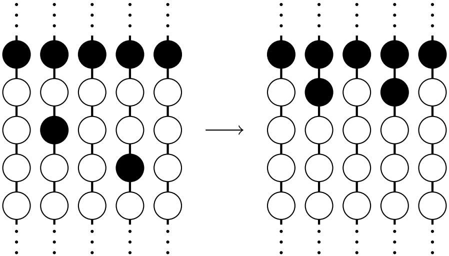

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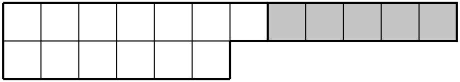

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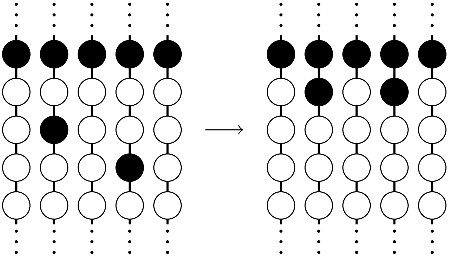

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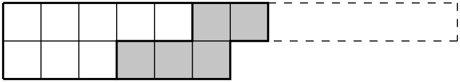

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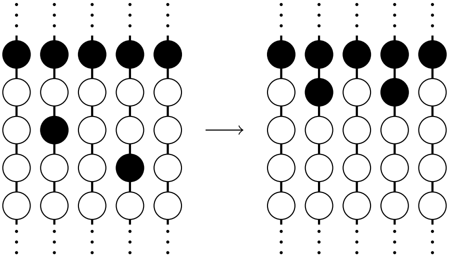

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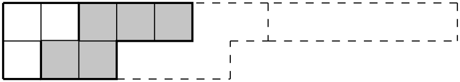

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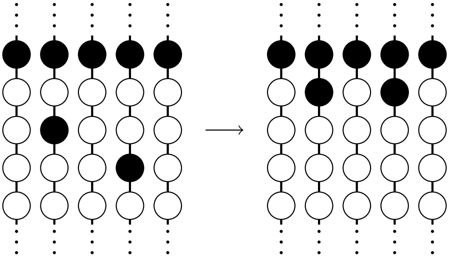

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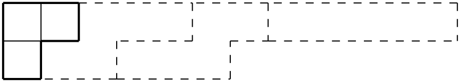

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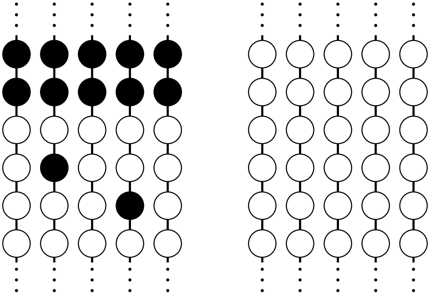
There are many combinatorial/algorithmic descriptions of the Mullineux map - by Mullineux (proved by Ford and Kleshchev), by Xu, by Brundan and Kujawa, by Jacon, by Fayers, and many others.

Abacus Mullineux algorithm



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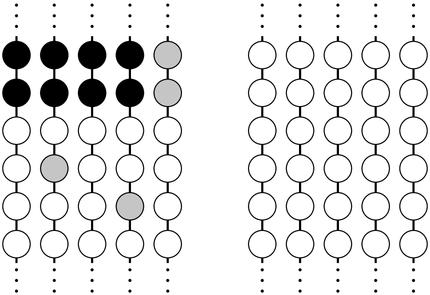

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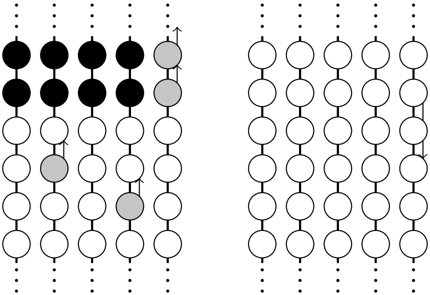

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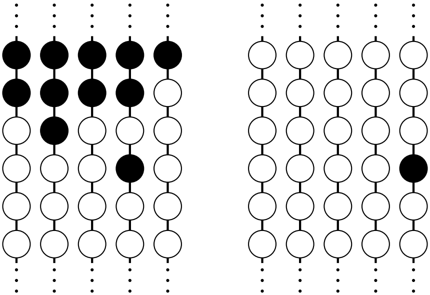

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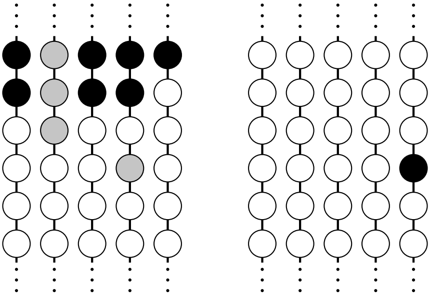

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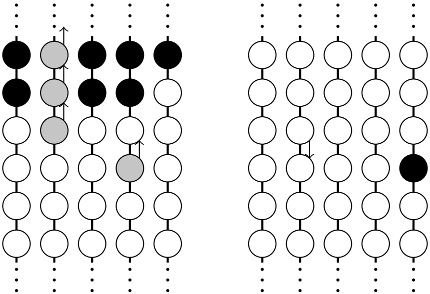

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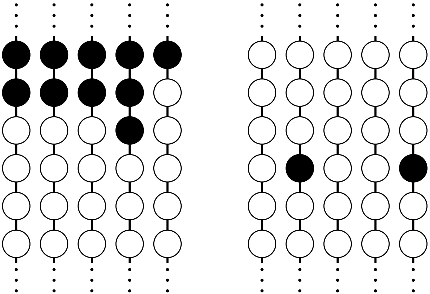

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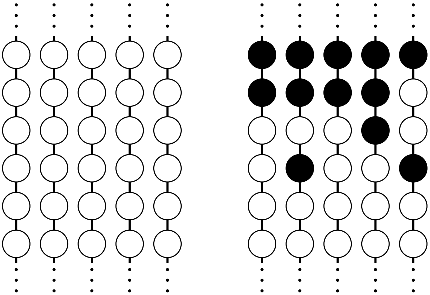

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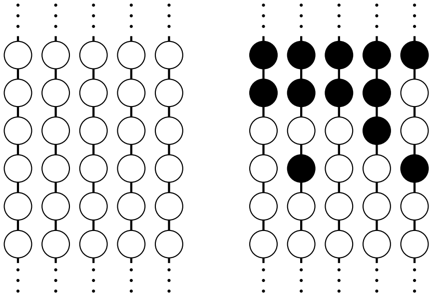

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1. Partitions
2. Mullineux map
3. Results
4. Work in progress

(p, d)-maximal and minimal partitions

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Main result

Theorem (T. 2024)

If λ is $(p, 2)$-maximal, then $\widetilde{m_p}(\lambda)$ is the $(p, 2)$-minimal partition with the same p-core as λ.

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Example: Recall that for $p = 5$ we have $E_2((2,1)) = \{(12,6),(10,8),(8,8,2),(8,6,4)\}$. Thus $\widetilde{m_p}((12,6)) = (8,6,4)$.

(p, d)-balanced partitions

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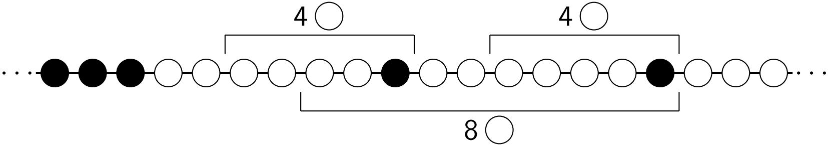

Figure: The partition $(12, 6)$ is $(5, 2)$-balanced.

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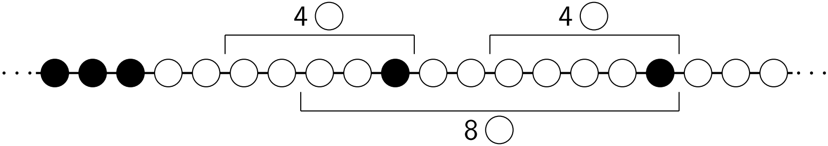

Figure: The partition $(12, 6)$ is $(5, 2)$-balanced.

Note that we only need to check ap positions preceded by an empty space.

(p, d)-shift-balanced partitions

Definition

A partition λ is (p, d)-shift-balanced if

- it is divisible by d, and
- for $a > 0$ and any $ap + 1$ consecutive positions in its James abacus ending in a bead, the number of empty places there is divisible by d.

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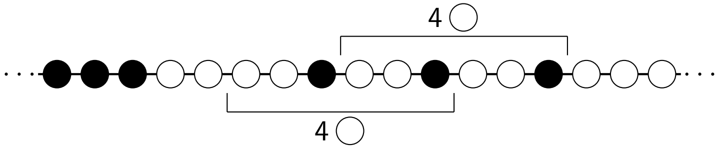

Figure: The partition $(8, 6, 4)$ is $(5, 2)$-shift-balanced.

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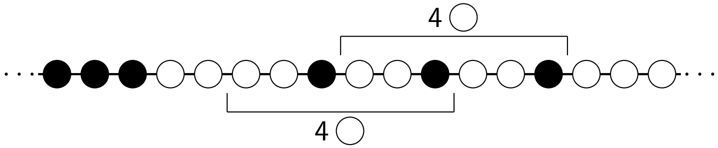

Figure: The partition $(8, 6, 4)$ is $(5, 2)$-shift-balanced.

Note that we only need to check $ap + 1$ positions starting with an empty space.

Ingredients

The main result then follows from the stronger statement.

Theorem (T. 2024)

The map $\widetilde{m_p}$ sends (p, d)-balanced partitions to (p, d)-shift-balanced partitions.

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Theorem (T. 2024)

The map $\widetilde{m_p}$ sends (p, d)-balanced partitions to (p, d)-shift-balanced partitions.

This is because for $d = 2$ the following holds.

Proposition

The $(p, 2)$–maximal (respectively $(p, 2)$-minimal) partitions are precisely the $(p, 2)$-balanced ($(p, 2)$-shift-balanced) partitions.

Outline

Unbalanced partitions

Definition

A partition λ is (p, d)-unbalanced if

- it is divisible by d, and
- for $a > 0$ and any ap consecutive positions in its James abacus ending in a bead *and preceded by an empty place*, the number of empty places there is *not* divisible by d.

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The (p, d)-maximal partitions and (p, d)-shift-unbalanced partitions coincide.

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It looks like the main theorem remains true for any d as long as we impose some restrictions on the underlying p-core.