Multiplicity-free induced characters of symmetric groups

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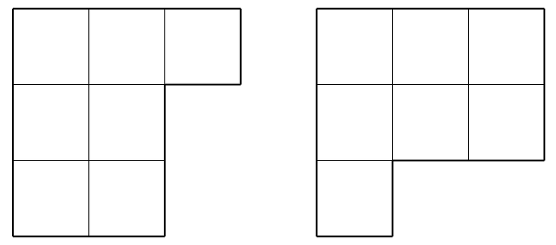
Introduction

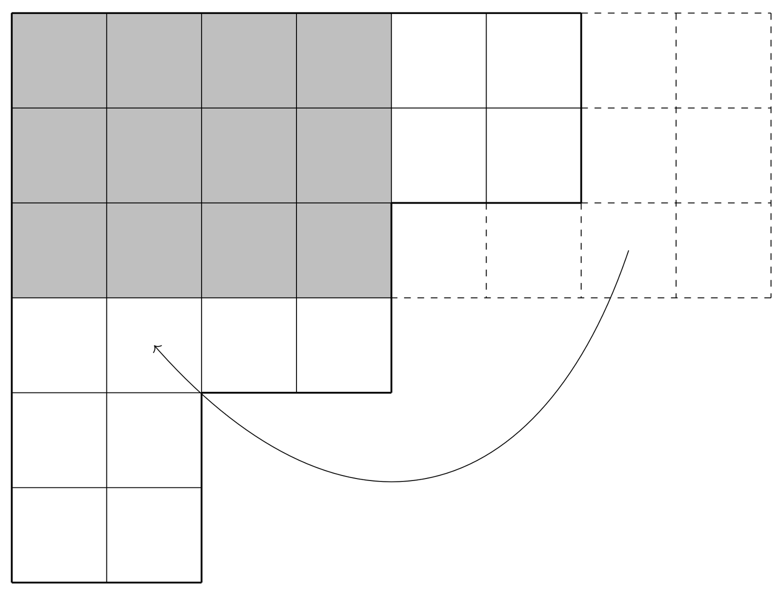
We say that a character of a finite group is *multiplicity-free* if all its irreducible constituents appear with multiplicity one. We also say that a character ρ of a subgroup of a symmetric group S_n is *induced-multiplicity-free* if $\rho\uparrow^{S_n}$ is multiplicity-free. Wildon in 2009 (for $n \geq 66$) and independently Godsil and Meagher in 2010 (for all n) classified the multiplicity-free permutation characters of symmetric groups S_n. In other words, they have found all subgroups $G \leq S_n$ such that the trivial character of G is induced-multiplicity-free. A more recent result by Anderson, Humphries and Nicholson found all subgroups $G \leq S_n$ such that any irreducible character of G is induced-multiplicity-free.

Our goal is to find all subgroups $G \leq S_n$ such that there is an induced-multiplicity-free character of G. We refer to such a subgroup as *multiplicity-free*. Crucial objects we need are *partitions* which label irreducible characters of symmetric groups. They are finite non-increasing sequences $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)$ of positive integers called parts. We use the standard notations a^b for b parts equal to a, $|\lambda|$ for the *size* of λ equal to $\sum_{i=1}^{t} \lambda_i$ and λ' for the conjugate partition of λ; see the Young diagrams of partitions $\lambda = (3, 2^2)$ and $\lambda' = (3^2, 1)$ of size 7 on the right.



Birectangular partitions

An interesting novel combinatorial question arises in the following setting. Suppose that $G \leq S_n$ and that $N := G \cap A_n \neq G$. Then for an irreducible induced-multiplicity-free character ρ of G such that $\rho \times \mathrm{sgn} \neq \rho$, the character $\rho\downarrow_N$ is irreducible and it is induced-multiplicity-free precisely when $\rho\uparrow^{S_n}$ does not have irreducible constituents labelled by conjugate partitions.



Let us take $G = S_m \wr S_2 \leq S_{2m}$ and $\rho = (\chi^{(a^b)})^{\widetilde{\times 2}}\chi^{\nu}$, a character labelled by partitions (a^b) of size m and ν of size 2. For $\nu = (2)$, the irreducible constituents of $\rho\uparrow^{S_{2m}}$ are labelled by (a,b)-birectangular partitions λ with $\lambda_1 + \lambda_2 + \dots + \lambda_b$ even, while for $\nu = (1^2)$ we want the sum to be odd. Here, (a,b)-*birectangular partitions* are partitions of size $2ab$ such that for all $1 \leq i \leq 2b$ we have $\lambda_i + \lambda_{2b+1-i} = 2a$; see a $(4,3)$-birectangular partition, which also appears in Proposition 1, on the left. The following result thus says when there is a pair of conjugate partitions which label constituents of $\rho\uparrow^{S_{2m}}$.

Proposition 1

Let $a > b$ be positive integers and write $d = a - b$. There is an (a,b)-birectangular partition λ such that λ' is also (a,b)-birectangular if and only if $d | a$. Moreover, in such a case there is a unique such λ given by $((2b)^{2d}, (2b-2d)^{2d}, \dots, (2d)^{2d})$.

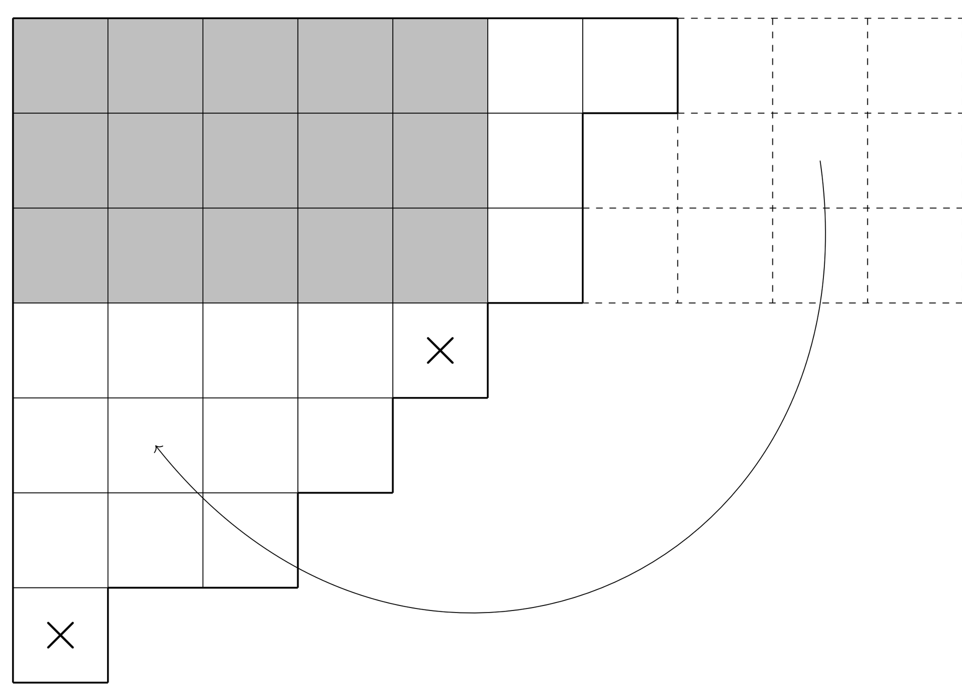
Additional boxes

In fact there is a 'stronger' version of Proposition 1 which is needed for other choices of ρ.

Proposition 2

Let t be a non-negative integer. Suppose that $a > b$ are positive integers. Write $d = a - b$ and $0 \leq r \leq d-1$ for the remainder of a modulo d. There is a partition λ of size $2ab + t$ such that for some (a,b)-birectangular partitions μ and ν we have $\mu, \nu' \subseteq \lambda$ if and only if $t \geq 2r(d - r)$.

A partition λ satisfying the constraints of Proposition 2 with $a = 5$, $b = 3$ and $t = 2$ is below.



Results

Our first main result is as follows.

Multiplicity-free subgroups

Suppose that $n \geq 66$. Multiplicity-free subgroups of S_n are given by certain subgroups of index 1, 2 and 4 of $S_k \times S_l, S_k \times S_m \wr S_2, S_k \times S_2 \wr S_h, S_m \wr S_3$ and $S_k \times L$ where L is one of $\mathrm{P\Gamma L}_2(\mathbb{F}_8) \leq S_9, \mathrm{ASL}_3(\mathbb{F}_2) \leq S_8, \mathrm{PGL}_2(\mathbb{F}_5) \leq S_6$ and $\mathrm{AGL}_1(\mathbb{F}_5) \leq S_5$ (where $k, l \geq 0, m \geq 2$ and $h \geq 3$).

The other main results then classify all irreducible induced-multiplicity-free characters apart from those induced from $A_k \times S_l$, $A_k \times A_l$ and $(S_k \times S_l) \cap A_{k+l}$. Here are some of the irreducible induced-multiplicity-free characters of $(S_m \wr S_2) \cap A_{2m}$.

Classification for $(S_m \wr S_2) \cap A_{2m}$

Let $m \geq 37$. The elementary irreducible induced-multiplicity-free characters of $G = (S_m \wr S_2) \cap A_{2m}$ are the constituents of $\left((\chi^{\mu})^{\widetilde{\times 2}}\chi^{\nu}\right)\downarrow_G$ with

(i) μ of the form (a^b) such that $a - b \nmid a$ or $\nu = (1^2)$,
(ii) μ of the form $(a+1, 1^b)$ with $b > a+1$,
(iii) μ of the form $(a+1, a^{b-1}), (a^b, 1)$ or $(a^{b-1}, a-1)$ with $a > b+2$ and $a - b \nmid a$,
(iv) μ of the form $((2b)^{b-1}, 2b-1), (2b+1, (2b)^{b-1})$ or $(3b+1, (3b)^{2b-1})$,
(v) μ of the form $(a^{a-1}, a-1)$ provided m is even.

One can use Proposition 1 to obtain (i). With a little more work one can recover (iii) from Proposition 2. To show that for $a - b \mid a$ or $a = b+2$ we do not get an induced-multiplicity-free character in (iii) (unless μ is as in (iv)) it suffices to provide two certain Littlewood–Richardson tableaux. Below is an example for $a = 4$, $b = 3$ and $\mu = (5, 4^2)$.