Multiplicity-free induced characters of symmetric groups

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Outline

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Example: Any irreducible character is multiplicity-free. The character of the regular representation of G is multiplicity-free if and only if G is abelian.

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Q2': For each multiplicity-free subgroup $G \leq S_n$ what are the irreducible G-multiplicity-free characters not-induced from $S_k \times A_l, A_k \times A_l$ or $(S_k \times S_l) \cap A_{k+l}$?

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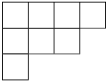

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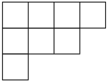

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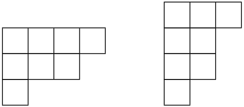

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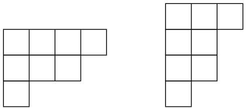

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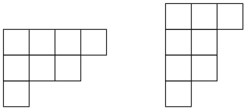

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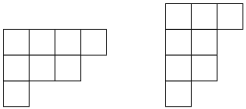

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There have been recent results regarding (strong) Gelfand pairs of the form $(M \wr S_h, G)$.

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The lower bound on $|G|$ also eliminates, for example, the diagonal subgroup of $S_k \times S_k$.

Non-transitive subgroups

From the following result multiplicity-free subgroups have at most two orbits.

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Let λ, μ and ν be non-empty partitions. Then $s_\lambda s_\mu s_\nu$ is not multiplicity-free.

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These results have consequences for non-elementary irreducible characters of wreath products.

Outline

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Example: Two standard λ/μ-tableaux with $\lambda = (4^2, 3, 2)$ and $\mu = (4, 2, 1)$. Their weights are $(3, 2, 1)$ and $(3, 1, 2)$, respectively.

$$\begin{ytableau} \none & \none & 1 & 1 \\ \none & 1 & 2 \\ 2 & 3 \end{ytableau} \qquad \begin{ytableau} \none & \none & 1 & 3 \\ \none & 1 & 2 \\ 1 & 3 \end{ytableau}$$

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			1

		1	3
	1	2	
1	3		

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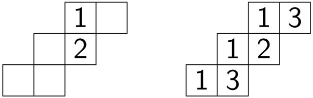

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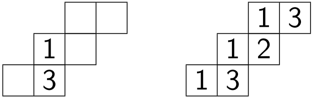

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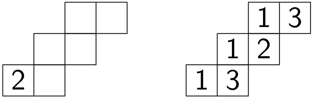

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Example: The reading words of above tableaux are 112132 and 312131.

Littlewood–Richardson rule

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Birectangular partitions

Proposition (T, 2023)

Let t be a non-negative integer. Suppose that $a > b$ are positive integers. Write $d = a - b$ and $0 \leq r \leq d - 1$ for the remainder of a modulo d. There is a partition λ of $2ab + t$ such that for some (a, b)-birectangular partitions μ and ν we have $\mu, \nu' \subseteq \lambda$ if and only if $t \geq 2r(d - r)$.

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					1	1
					2	
					3	
1	1	1	2	x		
2	2	2	3			
3	3	3				
x						

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			1	1	1	x
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4	5	5				
5						

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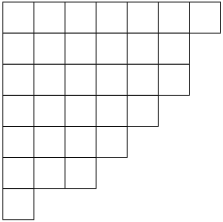

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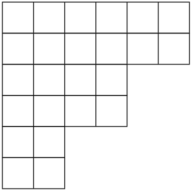

Outline

Multiplicity-free subgroups of S_n

Theorem (T, 2023)

Let $n \geq 66$. A subgroup $G \leq S_n$ is multiplicity-free if and only if it belongs to the list (throughout $k, l \geq 1$ and $m, h \geq 2$):

1. S_n and A_n,
2. $S_k \times S_l$, $(S_k \times S_l) \cap A_{k+l}$ and $A_k \times S_l$ with $k \neq 2$ and $A_k \times A_l$ with $k, l \neq 2$,
3. $S_k \times S_m \wr S_2$ and for $k \notin \{2m-3, 2m-2, 2m-1, 2m\}$ also groups $(S_k \times S_m \wr S_2) \cap A_{k+2m}$ and $T_{k,m,2}$,
4. $S_k \times S_2 \wr S_h$ and for $k \geq 2h+2$ also $(S_k \times S_2 \wr S_h) \cap A_{k+2h}$,
5. $S_m \wr S_2, (S_m \wr S_2) \cap A_{2m}, A_m \wr S_2$ and $T_{m,2}$,
6. $S_2 \wr S_h$ and $(S_2 \wr S_h) \cap A_{2h}$,
7. $S_m \wr S_3, (S_m \wr S_3) \cap A_{3m}$ and $T_{m,3}$,
8. $S_k \times L, A_k \times L$ and $(S_k \times L) \cap A_n$ where L is one of $\mathrm{P\Gamma L}(2,8)$, $\mathrm{ASL}(3,2)$, $\mathrm{PGL}(2,5)$ and $\mathrm{AGL}(1,5)$,
9. $A_k \times S_2 \wr S_2, N_k$ and $T_{k,2,h}$ with $h \in \{3,4\}$,
10. $A_m \wr S_2$ with $n = 2m+1$ provided m is a square.

Multiplicity-free subgroups of S_n

Theorem (T, 2023)

Let $n \geq 66$. A subgroup $G \leq S_n$ is multiplicity-free if and only if it belongs to the list (throughout $k, l \geq 1$ and $m, h \geq 2$):

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$(S_k \times S_2 \wr S_h)$-multiplicity-free characters

Instead of presenting the full answer to **Q2'** we only show the irreducible G-multiplicity-free characters for $G = S_k \times S_2 \wr S_h$ (with $k \geq 2$ and $h \geq 5$) and $G = (S_m \wr S_2) \cap A_{2m}$.

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Theorem (T, 2023)

Let $k \geq 2, h \geq 5$ and $G = S_k \times S_2 \wr S_h$. The irreducible G-multiplicity-free characters are $\chi^{(1^k)} \boxtimes \left((\chi^{(2)})^{\widetilde{\otimes h}}\chi^{(h)}\right)$ and $\chi^{(k)} \boxtimes \left((\chi^{(1^2)})^{\widetilde{\otimes h}}\chi^{(h)}\right)$.

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In the language of symmetric functions $s_\lambda(s_\nu \circ s_\mu)$ with $|\lambda| \geq 2 = |\mu|$ and $|\nu| \geq 5$ is multiplicity-free if and only if ν is a row partition and λ and μ are in some order a row and a column partition.

$((S_m \wr S_2) \cap A_{2m})$-multiplicity-free characters

Theorem (T, 2023)

Let $m \geq 37$ and $G = (S_m \wr S_2) \cap A_{2m}$. The elementary irreducible G-multiplicity-free characters are irreducible constituents of $\left((\chi^\mu)^{\widetilde{\otimes 2}} \chi^\nu\right) \downarrow_G^{S_m \wr S_2}$ with:

1. μ of the form (a^b) such that $a - b \nmid a$ or $\nu = (1^2)$;
2. μ of the form $(a+1, b)$ with $b > a+1$;
3. μ of the form $(a+1, a^{b-1}), (a^b, 1)$ or $(a^{b-1}, a-1)$ with $a > b+2$ and $a - b \nmid a$;
4. μ of the form $((2b)^{b-1}, 2b-1), (2b+1, (2b)^{b-1})$ or $(3b+1, (3b)^{2b-1})$;
5. μ of the form $(a^{a-1}, a-1)$ provided m is even.

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