Multiplicity-free induced characters of symmetric groups

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Q2: For each multiplicity-free subgroup $G \leq S_n$ what are the irreducible induced-multiplicity-free characters of G?

Multiplicity-free characters of S_n in literature

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- combined products $s_\lambda(s_\nu \circ s_\mu)$ - T. 2023.

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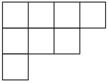

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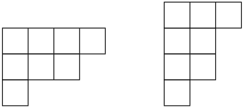

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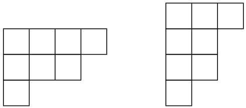

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The irreducible characters of $S_m \wr S_h$ are constructed from the *elementary irreducible characters* $\chi^\mu \wr \chi^\nu := (\chi^\mu)^{\widetilde{\times h}} \chi^\nu$ where $\mu \vdash m$ and $\nu \vdash h$.

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The lower bound on $|G|$ also eliminates, for example, the diagonal subgroup of $S_k \times S_k$.

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This impacts non-elementary irreducible induced-multiplicity-free characters of wreath products.

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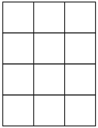

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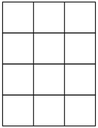

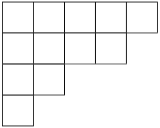

Birectangular partitions

Proposition (T, 2023)

Let $a > b$ be positive integers and write $d = a - b$. There is a partition λ such that λ and λ' are (a, b)-birectangular if and only if $d|a$. Moreover, in such a case $\lambda = ((2b)^{2d}, (2b-2d)^{2d}, \dots, (2d)^{2d})$.

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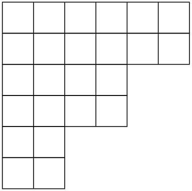
Example: The unique partition $\lambda = (6^2, 4^2, 2^2)$ such that λ and λ' are both $(4, 3)$-birectangular.



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- If ρ is an irreducible induced-multiplicity-free character of G, then (under mild conditions) $\rho\big\downarrow_N^G$ is induced-multiplicity-free if and only if there are no constituents χ^λ and $\chi^{\lambda'}$ of $\rho\big\uparrow_G^{S_n}$.

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- If $\rho = \chi^{(a^b)} \wr \chi^\nu$, then $\rho\big\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ is multiplicity-free with irreducible constituents labelled by (a, b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even for $\nu = (2)$, respectively, odd for $\nu = (1^2)$.

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Example: Let $G = S_{12} \wr S_2$ and $\rho = \chi^{(4^3)} \wr \chi^{(1^2)}$. Then $\rho\downarrow_N^G$ is induced-multiplicity-free.

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Example: Let $G = S_{12} \wr S_2$ and $\rho = \chi^{(4^3)} \wr \chi^{(2)}$. Then $\rho\big\downarrow_N^G$ is not induced-multiplicity-free.

Outline

Multiplicity-free subgroups of S_n (light version)

Theorem (T, 2023)

Suppose that $n \geq 66$. Multiplicity-free subgroups of S_n are given by certain subgroups of index $1, 2$ and 4 of

1. $S_k \times S_l$,
2. $S_k \times S_m \wr S_2$,
3. $S_k \times S_2 \wr S_h$,
4. $S_m \wr S_3$,
5. $S_k \times L$ where L is one of $\mathrm{P\Gamma L}_2(\mathbb{F}_8) \leq S_9$, $\mathrm{ASL}_3(\mathbb{F}_2) \leq S_8$, $\mathrm{PGL}_2(\mathbb{F}_5) \leq S_6$ and $\mathrm{AGL}_1(\mathbb{F}_5) \leq S_5$,

where $k, l \geq 0$, $m \geq 2$ and $h \geq 3$.

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where $k, l \geq 0$, $m \geq 2$ and $h \geq 3$.

The trivial characters of groups in (2), (3) and (4) are not induced-multiplicity-free, provided $k \geq 2$.

Multiplicity-free subgroups of S_n (full version)

Theorem (T, 2023)

Let $n \geq 66$. A subgroup $G \leq S_n$ is multiplicity-free if and only if it belongs to the list (throughout $k, l \geq 1$, $m \geq 2$ and $h \geq 3$):

1. S_n and A_n,
2. $S_k \times S_l$, $(S_k \times S_l) \cap A_{k+l}$ and $A_k \times S_l$ with $k \neq 2$ and $A_k \times A_l$ with $k, l \neq 2$,
3. $S_k \times S_m \wr S_2$ and for $k \notin \{2m-3, 2m-2, 2m-1, 2m\}$ also groups $(S_k \times S_m \wr S_2) \cap A_{k+2m}$ and $T_{k,m,2}$,
4. $S_k \times S_2 \wr S_h$ and for $k \geq 2h+2$ also $(S_k \times S_2 \wr S_h) \cap A_{k+2h}$,
5. $S_m \wr S_2, (S_m \wr S_2) \cap A_{2m}, A_m \wr S_2$ and $T_{m,2}$,
6. $S_2 \wr S_h$ and $(S_2 \wr S_h) \cap A_{2h}$,
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8. $S_k \times L, A_k \times L$ and $(S_k \times L) \cap A_n$ where L is one of $\mathrm{P\Gamma L}(2,8), \mathrm{ASL}(3,2)$, $\mathrm{PGL}(2,5)$ and $\mathrm{AGL}(1,5)$,
9. $A_k \times S_2 \wr S_2, N_k$ and $T_{k,2,h}$ with $h \in \{3,4\}$,
10. $A_m \wr S_2$ with $n = 2m+1$ provided m is a square.

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Induced-multiplicity-free characters of $(S_m \wr S_2) \cap A_{2m}$

Theorem (T, 2023)

Let $m \geq 37$. The elementary irreducible induced-multiplicity-free characters of $G = (S_m \wr S_2) \cap A_{2m}$ are the irreducible constituents of $(\chi^\mu \wr \chi^\nu) \downarrow_G^{S_m \wr S_2}$ with:

1. μ of the form (a^b) such that $a - b \nmid a$ or $\nu = (1^2)$;
2. μ of the form $(a+1, b)$ with $b > a+1$;
3. μ of the form $(a+1, a^{b-1}), (a^b, 1)$ or $(a^{b-1}, a-1)$ with $a > b+2$ and $a - b \nmid a$;
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