Multiplicity-free induced characters of symmetric groups

Pavel Turek

Royal Holloway, University of London

July 11, 2023

Outline

Outline

Multiplicity-free characters

Let G be a (finite) group and V a finite-dimensional $\mathbb{C}G$-module.

Multiplicity-free characters

Let G be a (finite) group and V a finite-dimensional $\mathbb{C}G$-module. The character of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the matrices representing the G-action on V.

Multiplicity-free characters

Let G be a (finite) group and V a finite-dimensional $\mathbb{C}G$-module. The character of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the matrices representing the G-action on V. We adapt the following notions to characters:

Multiplicity-free characters

Let G be a (finite) group and V a finite-dimensional $\mathbb{C}G$-module. The character of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the matrices representing the G-action on V. We adapt the following notions to characters:

- direct sum and tensor products,

Multiplicity-free characters

Let G be a (finite) group and V a finite-dimensional $\mathbb{C}G$-module. The character of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the matrices representing the G-action on V. We adapt the following notions to characters:

- direct sum and tensor products,
- irreducibility and decomposition,

Multiplicity-free characters

Let G be a (finite) group and V a finite-dimensional $\mathbb{C}G$-module. The character of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the matrices representing the G-action on V. We adapt the following notions to characters:

- direct sum and tensor products,
- irreducibility and decomposition,
- induction and reduction.

Multiplicity-free characters

Let G be a (finite) group and V a finite-dimensional $\mathbb{C}G$-module. The character of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the matrices representing the G-action on V. We adapt the following notions to characters:

- direct sum and tensor products,
- irreducibility and decomposition,
- induction and reduction.

We say that a character is multiplicity-free if it has no irreducible constituents of multiplicity more than one.

Multiplicity-free characters

Let G be a (finite) group and V a finite-dimensional $\mathbb{C}G$-module. The character of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the matrices representing the G-action on V. We adapt the following notions to characters:

- direct sum and tensor products,
- irreducibility and decomposition,
- induction and reduction.

We say that a character is multiplicity-free if it has no irreducible constituents of multiplicity more than one.

Example: Any irreducible character is multiplicity-free.

Multiplicity-free characters

Let G be a (finite) group and V a finite-dimensional $\mathbb{C}G$-module. The character of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the matrices representing the G-action on V. We adapt the following notions to characters:

- direct sum and tensor products,
- irreducibility and decomposition,
- induction and reduction.

We say that a character is multiplicity-free if it has no irreducible constituents of multiplicity more than one.

Example: Any irreducible character is multiplicity-free. The character of the regular representation of G is multiplicity-free if and only if G is abelian.

Setting

Fix a non-negative integer n.

Setting

Fix a non-negative integer n. Given a character ρ of a subgroup $G \leq S_n$, we say ρ is induced-multiplicity-free if $\rho\big\uparrow_G^{S_n}$ is multiplicity-free.

Setting

Fix a non-negative integer n. Given a character ρ of a subgroup $G \leq S_n$, we say ρ is induced-multiplicity-free if $\rho\big\uparrow_G^{S_n}$ is multiplicity-free. A subgroup $G \leq S_n$ is multiplicity-free if there is an induced-multiplicity-free character of G.

Setting

Fix a non-negative integer n. Given a character ρ of a subgroup $G \leq S_n$, we say ρ is induced-multiplicity-free if $\rho\uparrow_G^{S_n}$ is multiplicity-free. A subgroup $G \leq S_n$ is multiplicity-free if there is an induced-multiplicity-free character of G.

Example: The groups S_n and A_n are multiplicity-free.

Setting

Fix a non-negative integer n. Given a character ρ of a subgroup $G \leq S_n$, we say ρ is induced-multiplicity-free if $\rho\uparrow_G^{S_n}$ is multiplicity-free. A subgroup $G \leq S_n$ is multiplicity-free if there is an induced-multiplicity-free character of G.

Example: The groups S_n and A_n are multiplicity-free. The trivial group is multiplicity-free if and only if $n \leq 2$.

Setting

Fix a non-negative integer n. Given a character ρ of a subgroup $G \leq S_n$, we say ρ is induced-multiplicity-free if $\rho\uparrow_G^{S_n}$ is multiplicity-free. A subgroup $G \leq S_n$ is multiplicity-free if there is an induced-multiplicity-free character of G.

Example: The groups S_n and A_n are multiplicity-free. The trivial group is multiplicity-free if and only if $n \leq 2$.

Q1: What are the multiplicity-free subgroups of S_n?

Setting

Fix a non-negative integer n. Given a character ρ of a subgroup $G \leq S_n$, we say ρ is induced-multiplicity-free if $\rho\uparrow_G^{S_n}$ is multiplicity-free. A subgroup $G \leq S_n$ is multiplicity-free if there is an induced-multiplicity-free character of G.

Example: The groups S_n and A_n are multiplicity-free. The trivial group is multiplicity-free if and only if $n \leq 2$.

Q1: What are the multiplicity-free subgroups of S_n?
Q2: For each multiplicity-free subgroup $G \leq S_n$ what are the irreducible induced-multiplicity-free characters of G?

Characters of symmetric groups

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n.

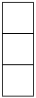
Characters of symmetric groups

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n. We emphasise the identity $\chi^\lambda \times \mathrm{sgn} = \chi^{\lambda'}$, where λ' is the conjugate partition of λ, for a later use.

Characters of symmetric groups

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n. We emphasise the identity $\chi^\lambda \times \mathrm{sgn} = \chi^{\lambda'}$, where λ' is the conjugate partition of λ, for a later use.

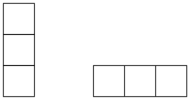
Example: The group S_3 has three irreducible characters. They are the sign character $\chi^{(1^3)}$



Characters of symmetric groups

The irreducible characters of the symmetric group S_n, commonly denoted by χ^{λ}, are labelled by partitions λ of n. We emphasise the identity $\chi^{\lambda} \times \mathrm{sgn} = \chi^{\lambda'}$, where λ' is the conjugate partition of λ, for a later use.

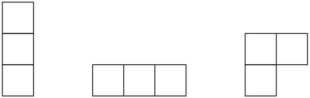
Example: The group S_3 has three irreducible characters. They are the sign character $\chi^{(1^3)}$, the trivial character $\chi^{(3)}$



Characters of symmetric groups

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n. We emphasise the identity $\chi^\lambda \times \mathrm{sgn} = \chi^{\lambda'}$, where λ' is the conjugate partition of λ, for a later use.

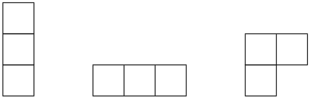
Example: The group S_3 has three irreducible characters. They are the sign character $\chi^{(1^3)}$, the trivial character $\chi^{(3)}$ and the non-trivial constituent $\chi^{(2,1)}$ of the natural permutation character.



Characters of symmetric groups

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n. We emphasise the identity $\chi^\lambda \times \mathrm{sgn} = \chi^{\lambda'}$, where λ' is the conjugate partition of λ, for a later use.

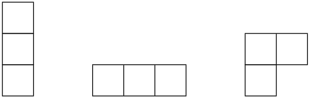
Example: The group S_3 has three irreducible characters. They are the sign character $\chi^{(1^3)}$, the trivial character $\chi^{(3)}$ and the non-trivial constituent $\chi^{(2,1)}$ of the natural permutation character. Observe $\chi^{(2,1)} \times \mathrm{sgn} = \chi^{(2,1)}$.



Characters of symmetric groups

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n. We emphasise the identity $\chi^\lambda \times \mathrm{sgn} = \chi^{\lambda'}$, where λ' is the conjugate partition of λ, for a later use.

Example: The group S_3 has three irreducible characters. They are the sign character $\chi^{(1^3)}$, the trivial character $\chi^{(3)}$ and the non-trivial constituent $\chi^{(2,1)}$ of the natural permutation character. Observe $\chi^{(2,1)} \times \mathrm{sgn} = \chi^{(2,1)}$.

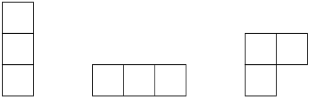


For $S_k \times S_l$ we then have irreducible characters $\chi^\mu \boxtimes \chi^\nu$ where $\mu \vdash k$ and $\nu \vdash l$.

Characters of symmetric groups

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n. We emphasise the identity $\chi^\lambda \times \mathrm{sgn} = \chi^{\lambda'}$, where λ' is the conjugate partition of λ, for a later use.

Example: The group S_3 has three irreducible characters. They are the sign character $\chi^{(1^3)}$, the trivial character $\chi^{(3)}$ and the non-trivial constituent $\chi^{(2,1)}$ of the natural permutation character. Observe $\chi^{(2,1)} \times \mathrm{sgn} = \chi^{(2,1)}$.



For $S_k \times S_l$ we then have irreducible characters $\chi^\mu \boxtimes \chi^\nu$ where $\mu \vdash k$ and $\nu \vdash l$. And for wreath products $S_m \wr S_h$ we construct all irreducible characters from the elementary irreducible characters $(\chi^\mu)^{\widetilde{\times h}}\chi^\nu$ where $\mu \vdash m$ and $\nu \vdash h$.

Multiplicity-free characters of S_n in literature

Multiplicity-free permutation characters of S_n were found by Wildon in 2009 (for $n \geq 66$) and independently also by Godsil and Meagher in 2010.

Multiplicity-free characters of S_n in literature

Multiplicity-free permutation characters of S_n were found by Wildon in 2009 (for $n \geq 66$) and independently also by Godsil and Meagher in 2010. A classification of subgroups $G \leq S_n$ such that all the irreducible characters of G are induced-multiplicity-free was established by Anderson, Humphries and Nicholson in 2021.

Multiplicity-free characters of S_n in literature

Multiplicity-free permutation characters of S_n were found by Wildon in 2009 (for $n \geq 66$) and independently also by Godsil and Meagher in 2010. A classification of subgroups $G \leq S_n$ such that all the irreducible characters of G are induced-multiplicity-free was established by Anderson, Humphries and Nicholson in 2021.

All these results can be rephrased using (strong) Gelfand pairs.

Multiplicity-free characters of S_n in literature

Multiplicity-free permutation characters of S_n were found by Wildon in 2009 (for $n \geq 66$) and independently also by Godsil and Meagher in 2010. A classification of subgroups $G \leq S_n$ such that all the irreducible characters of G are induced-multiplicity-free was established by Anderson, Humphries and Nicholson in 2021.

All these results can be rephrased using (strong) Gelfand pairs. Recall that a pair of groups (H, G) with $G \leq H$ is a Gelfand pair if the permutation character of H acting on the cosets of G is multiplicity-free. It is a strong Gelfand pair if all irreducible characters of G stay multiplicity-free after applying the induction to H.

Multiplicity-free characters of S_n in literature

Multiplicity-free permutation characters of S_n were found by Wildon in 2009 (for $n \geq 66$) and independently also by Godsil and Meagher in 2010. A classification of subgroups $G \leq S_n$ such that all the irreducible characters of G are induced-multiplicity-free was established by Anderson, Humphries and Nicholson in 2021.

All these results can be rephrased using (strong) Gelfand pairs. Recall that a pair of groups (H, G) with $G \leq H$ is a Gelfand pair if the permutation character of H acting on the cosets of G is multiplicity-free. It is a strong Gelfand pair if all irreducible characters of G stay multiplicity-free after applying the induction to H. Our **Q1** asks for 'weak Gelfand pairs' (S_n, G).

Symmetric functions

The characters of symmetric functions are linked to the symmetric functions via the following dictionary:

Symmetric functions

The characters of symmetric functions are linked to the symmetric functions via the following dictionary:

- $\chi^\lambda \mapsto$ Schur function s_λ,

Symmetric functions

The characters of symmetric functions are linked to the symmetric functions via the following dictionary:

- $\chi^\lambda \mapsto$ Schur function s_λ,
- $(\chi^\mu \boxtimes \chi^\nu)\uparrow_{S_k \times S_l}^{S_{k+l}} \mapsto s_\mu s_\nu$,

Symmetric functions

The characters of symmetric functions are linked to the symmetric functions via the following dictionary:

- $\chi^\lambda \mapsto$ Schur function s_λ,
- $(\chi^\mu \boxtimes \chi^\nu) \big\uparrow_{S_k \times S_l}^{S_{k+l}} \mapsto s_\mu s_\nu$,
- $\left((\chi^\mu)^{\widetilde{\times h}} \chi^\nu\right) \big\uparrow_{S_m \wr S_h}^{S_{mh}} \mapsto$ plethysm $s_\nu \circ s_\mu$.

Symmetric functions

The characters of symmetric functions are linked to the symmetric functions via the following dictionary:

- $\chi^{\lambda} \mapsto$ Schur function s_{λ},
- $(\chi^{\mu} \boxtimes \chi^{\nu}) \big\uparrow_{S_k \times S_l}^{S_{k+l}} \mapsto s_{\mu} s_{\nu}$,
- $\left((\chi^{\mu})^{\widetilde{\times h}} \chi^{\nu}\right) \big\uparrow_{S_m \wr S_h}^{S_{mh}} \mapsto$ plethysm $s_{\nu} \circ s_{\mu}$.

Stembridge in 2001 classified the multiplicity-free products of Schur functions.

Symmetric functions

The characters of symmetric functions are linked to the symmetric functions via the following dictionary:

- $\chi^\lambda \mapsto$ Schur function s_λ,
- $(\chi^\mu \boxtimes \chi^\nu) \big\uparrow_{S_k \times S_l}^{S_{k+l}} \mapsto s_\mu s_\nu$,
- $\left((\chi^\mu)^{\widetilde{\times h}} \chi^\nu\right) \big\uparrow_{S_m \wr S_h}^{S_{mh}} \mapsto$ plethysm $s_\nu \circ s_\mu$.

Stembridge in 2001 classified the multiplicity-free products of Schur functions. Recently Bessenrodt, Bowman and Paget found the multiplicity-free plethysms $s_\nu \circ s_\mu$.

Symmetric functions

The characters of symmetric functions are linked to the symmetric functions via the following dictionary:

- $\chi^\lambda \mapsto$ Schur function s_λ,
- $(\chi^\mu \boxtimes \chi^\nu) \big\uparrow_{S_k \times S_l}^{S_{k+l}} \mapsto s_\mu s_\nu$,
- $\left((\chi^\mu)^{\widetilde{\times h}} \chi^\nu\right) \big\uparrow_{S_m \wr S_h}^{S_{mh}} \mapsto$ plethysm $s_\nu \circ s_\mu$.

Stembridge in 2001 classified the multiplicity-free products of Schur functions. Recently Bessenrodt, Bowman and Paget found the multiplicity-free plethysms $s_\nu \circ s_\mu$.

As a consequence of our work a classification of multiplicity-free symmetric functions $s_\lambda(s_\nu \circ s_\mu)$ is found.

Symmetric functions

The characters of symmetric functions are linked to the symmetric functions via the following dictionary:

- $\chi^\lambda \mapsto$ Schur function s_λ,
- $(\chi^\mu \boxtimes \chi^\nu) \big\uparrow_{S_k \times S_l}^{S_{k+l}} \mapsto s_\mu s_\nu$,
- $\left((\chi^\mu)^{\widetilde{\times h}} \chi^\nu\right) \big\uparrow_{S_m \wr S_h}^{S_{mh}} \mapsto$ plethysm $s_\nu \circ s_\mu$.

Stembridge in 2001 classified the multiplicity-free products of Schur functions. Recently Bessenrodt, Bowman and Paget found the multiplicity-free plethysms $s_\nu \circ s_\mu$.

As a consequence of our work a classification of multiplicity-free symmetric functions $s_\lambda(s_\nu \circ s_\mu)$ is found. Moreover, for n sufficiently large, one can observe this is the only way to 'compose' 3 or more Schur functions and get a multiplicity-free symmetric function.

Outline

Small subgroups

Any multiplicity-free character of S_n has degree at most $a_n := \sum_{\lambda \vdash n} \chi^\lambda(1)$.

Small subgroups

Any multiplicity-free character of S_n has degree at most $a_n := \sum_{\lambda \vdash n} \chi^\lambda(1)$. Consequently, any multiplicity-free subgroup G must satisfy $|G| \geq n!/a_n$.

Small subgroups

Any multiplicity-free character of S_n has degree at most $a_n := \sum_{\lambda \vdash n} \chi^\lambda(1)$.
Consequently, any multiplicity-free subgroup G must satisfy $|G| \geq n!/a_n$.
It can be shown that a_n counts the involutions of S_n.

Small subgroups

Any multiplicity-free character of S_n has degree at most $a_n := \sum_{\lambda \vdash n} \chi^\lambda(1)$. Consequently, any multiplicity-free subgroup G must satisfy $|G| \geq n!/a_n$. It can be shown that a_n counts the involutions of S_n. In turn for $n \geq 7$ we have $a_n < n!/\left(2\lceil \frac{n}{2} \rceil!\right)$ and for $n \geq 11$ also $a_n < n!/(2^{n-1})$.

Small subgroups

Any multiplicity-free character of S_n has degree at most $a_n := \sum_{\lambda \vdash n} \chi^\lambda(1)$. Consequently, any multiplicity-free subgroup G must satisfy $|G| \geq n!/a_n$. It can be shown that a_n counts the involutions of S_n. In turn for $n \geq 7$ we have $a_n < n!/\left(2\lceil \frac{n}{2} \rceil !\right)$ and for $n \geq 11$ also $a_n < n!/(2^{n-1})$. Thus we eliminate almost all primitive subgroups using the following.

Theorem (Maróti, 2002)

Let G be a primitive group of S_n which is not S_n or A_n. Then the order of G is less or equal to 2^{n-1} or $n \leq 24$.

Small subgroups

Any multiplicity-free character of S_n has degree at most $a_n := \sum_{\lambda \vdash n} \chi^\lambda(1)$. Consequently, any multiplicity-free subgroup G must satisfy $|G| \geq n!/a_n$. It can be shown that a_n counts the involutions of S_n. In turn for $n \geq 7$ we have $a_n < n!/\left(2\lceil \frac{n}{2} \rceil !\right)$ and for $n \geq 11$ also $a_n < n!/(2^{n-1})$. Thus we eliminate almost all primitive subgroups using the following.

Theorem (Maróti, 2002)

Let G be a primitive group of S_n which is not S_n or A_n. Then the order of G is less or equal to 2^{n-1} or $n \leq 24$.

The lower bound on $|G|$, for example, also eliminates the diagonal subgroup of $S_k \times S_k$.

Non-transitive subgroups

From the following result multiplicity-free subgroups have at most two orbits.

Theorem (Stembridge, 2001)

Let λ, μ and ν be non-empty partitions. Then $s_\lambda s_\mu s_\nu$ is not multiplicity-free.

Non-transitive subgroups

From the following result multiplicity-free subgroups have at most two orbits.

Theorem (Stembridge, 2001)

Let λ, μ and ν be non-empty partitions. Then $s_\lambda s_\mu s_\nu$ is not multiplicity-free.

With a little bit more work I have shown that multiplicity-free subgroups $K \times L \leq S_k \times S_l$ have K or L equal to the respective symmetric or alternating group.

Non-transitive subgroups

From the following result multiplicity-free subgroups have at most two orbits.

Theorem (Stembridge, 2001)

Let λ, μ and ν be non-empty partitions. Then $s_\lambda s_\mu s_\nu$ is not multiplicity-free.

With a little bit more work I have shown that multiplicity-free subgroups $K \times L \leq S_k \times S_l$ have K or L equal to the respective symmetric or alternating group. Moreover if $K = A_k$, then $l \leq 9$.

Non-transitive subgroups

From the following result multiplicity-free subgroups have at most two orbits.

Theorem (Stembridge, 2001)

Let λ, μ and ν be non-empty partitions. Then $s_\lambda s_\mu s_\nu$ is not multiplicity-free.

With a little bit more work I have shown that multiplicity-free subgroups $K \times L \leq S_k \times S_l$ have K or L equal to the respective symmetric or alternating group. Moreover if $K = A_k$, then $l \leq 9$.

These results impose restrictions on non-elementary irreducible induced-multiplicity-free characters of wreath products.

Outline

Index two subgroups

Suppose that $G \neq N := G \cap A_n$.

Index two subgroups

Suppose that $G \neq N := G \cap A_n$. The irreducible characters of N are constituents of restrictions of irreducible characters of G.

Index two subgroups

Suppose that $G \neq N := G \cap A_n$. The irreducible characters of N are constituents of restrictions of irreducible characters of G. In particular, if ρ is an irreducible character of G and

- $\rho \times \mathrm{sgn} = \rho$, then $\rho\big\downarrow_N^G$ has two irreducible constituents

Index two subgroups

Suppose that $G \neq N := G \cap A_n$. The irreducible characters of N are constituents of restrictions of irreducible characters of G. In particular, if ρ is an irreducible character of G and

- $\rho \times \mathrm{sgn} = \rho$, then $\rho\big\downarrow_N^G$ has two irreducible constituents and if κ is one of them, then $\kappa\big\uparrow_N^{S_n} = \rho\big\uparrow_G^{S_n}$;

Index two subgroups

Suppose that $G \neq N := G \cap A_n$. The irreducible characters of N are constituents of restrictions of irreducible characters of G. In particular, if ρ is an irreducible character of G and

- $\rho \times \mathrm{sgn} = \rho$, then $\rho\downarrow_N^G$ has two irreducible constituents and if κ is one of them, then $\kappa\uparrow_N^{S_n} = \rho\uparrow_G^{S_n}$;
- $\rho \times \mathrm{sgn} \neq \rho$, then $\kappa := \rho\downarrow_N^G$ is irreducible

Index two subgroups

Suppose that $G \neq N := G \cap A_n$. The irreducible characters of N are constituents of restrictions of irreducible characters of G. In particular, if ρ is an irreducible character of G and

- $\rho \times \mathrm{sgn} = \rho$, then $\rho\big\downarrow_N^G$ has two irreducible constituents and if κ is one of them, then $\kappa\big\uparrow_N^{S_n} = \rho\big\uparrow_G^{S_n}$;
- $\rho \times \mathrm{sgn} \neq \rho$, then $\kappa := \rho\big\downarrow_N^G$ is irreducible and $\kappa\big\uparrow_N^{S_n} = \rho\big\uparrow_G^{S_n} + \rho\big\uparrow_G^{S_n} \times \mathrm{sgn}$.

Index two subgroups

Suppose that $G \neq N := G \cap A_n$. The irreducible characters of N are constituents of restrictions of irreducible characters of G. In particular, if ρ is an irreducible character of G and

- $\rho \times \mathrm{sgn} = \rho$, then $\rho\big\downarrow_N^G$ has two irreducible constituents and if κ is one of them, then $\kappa\big\uparrow_N^{S_n} = \rho\big\uparrow_G^{S_n}$;
- $\rho \times \mathrm{sgn} \neq \rho$, then $\kappa := \rho\big\downarrow_N^G$ is irreducible and $\kappa\big\uparrow_N^{S_n} = \rho\big\uparrow_G^{S_n} + \rho\big\uparrow_G^{S_n} \times \mathrm{sgn}$.

In the first case κ is induced-multiplicity-free if and only if ρ is induced-multiplicity-free.

Index two subgroups

Suppose that $G \neq N := G \cap A_n$. The irreducible characters of N are constituents of restrictions of irreducible characters of G. In particular, if ρ is an irreducible character of G and

- $\rho \times \mathrm{sgn} = \rho$, then $\rho\big\downarrow_N^G$ has two irreducible constituents and if κ is one of them, then $\kappa\big\uparrow_N^{S_n} = \rho\big\uparrow_G^{S_n}$;
- $\rho \times \mathrm{sgn} \neq \rho$, then $\kappa := \rho\big\downarrow_N^G$ is irreducible and $\kappa\big\uparrow_N^{S_n} = \rho\big\uparrow_G^{S_n} + \rho\big\uparrow_G^{S_n} \times \mathrm{sgn}$.

In the first case κ is induced-multiplicity-free if and only if ρ is induced-multiplicity-free. In the second case we also need $\rho\big\uparrow_G^{S_n}$ not to have constituents of the form χ^λ and $\chi^{\lambda'}$ for some λ.

Index two subgroups

Suppose that $G \neq N := G \cap A_n$. The irreducible characters of N are constituents of restrictions of irreducible characters of G. In particular, if ρ is an irreducible character of G and

- $\rho \times \mathrm{sgn} = \rho$, then $\rho\big\downarrow_N^G$ has two irreducible constituents and if κ is one of them, then $\kappa\big\uparrow_N^{S_n} = \rho\big\uparrow_G^{S_n}$;
- $\rho \times \mathrm{sgn} \neq \rho$, then $\kappa := \rho\big\downarrow_N^G$ is irreducible and $\kappa\big\uparrow_N^{S_n} = \rho\big\uparrow_G^{S_n} + \rho\big\uparrow_G^{S_n} \times \mathrm{sgn}$.

In the first case κ is induced-multiplicity-free if and only if ρ is induced-multiplicity-free. In the second case we also need $\rho\big\uparrow_G^{S_n}$ not to have constituents of the form χ^λ and $\chi^{\lambda'}$ for some λ.

Example: Let $G = S_2 \times S_3$. Since $\left(\chi^{(2)} \boxtimes \chi^{(2,1)}\right)\big\uparrow_G^{S_5}$ decomposes as $\chi^{(4,1)} + \chi^{(3,2)} + \chi^{(3,1^2)} + \chi^{(2^2,1)}$, the character $\left(\chi^{(2)} \boxtimes \chi^{(2,1)}\right)\big\downarrow_N^G$ is not induced-multiplicity-free.

Index two subgroups

Suppose that $G \neq N := G \cap A_n$. The irreducible characters of N are constituents of restrictions of irreducible characters of G. In particular, if ρ is an irreducible character of G and

- $\rho \times \mathrm{sgn} = \rho$, then $\rho\big\downarrow_N^G$ has two irreducible constituents and if κ is one of them, then $\kappa\big\uparrow_N^{S_n} = \rho\big\uparrow_G^{S_n}$;
- $\rho \times \mathrm{sgn} \neq \rho$, then $\kappa := \rho\big\downarrow_N^G$ is irreducible and $\kappa\big\uparrow_N^{S_n} = \rho\big\uparrow_G^{S_n} + \rho\big\uparrow_G^{S_n} \times \mathrm{sgn}$.

In the first case κ is induced-multiplicity-free if and only if ρ is induced-multiplicity-free. In the second case we also need $\rho\big\uparrow_G^{S_n}$ not to have constituents of the form χ^λ and $\chi^{\lambda'}$ for some λ.

Example: Let $G = S_2 \times S_3$. Since $\left(\chi^{(2)} \boxtimes \chi^{(2,1)}\right)\big\uparrow_G^{S_5}$ decomposes as $\chi^{(4,1)} + \chi^{(3,2)} + \chi^{(3,1^2)} + \chi^{(2^2,1)}$, the character $\left(\chi^{(2)} \boxtimes \chi^{(2,1)}\right)\big\downarrow_N^G$ is not induced-multiplicity-free.

Formula for $\left((\chi^{(a^b)})^{\widetilde{\times 2}}\chi^{\nu}\right)\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$

For $a, b \in \mathbb{N}$ we say that a partition λ of size $2ab$ is (a, b)-birectangular if for all $1 \leq i \leq 2b$ the equality $\lambda_i + \lambda_{2b+1-i} = 2a$ holds true.

Formula for $\left((\chi^{(a^b)})^{\widetilde{\times 2}}\chi^{\nu}\right)\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$

For $a, b \in \mathbb{N}$ we say that a partition λ of size $2ab$ is (a, b)-birectangular if for all $1 \leq i \leq 2b$ the equality $\lambda_i + \lambda_{2b+1-i} = 2a$ holds true.

Proposition

The character $\left((\chi^{(a^b)})^{\widetilde{\times 2}}\chi^{\nu}\right)\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ with $\nu = (2)$, respectively, $\nu = (1^2)$ is multiplicity-free with constituents labelled by (a, b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even, respectively, odd.

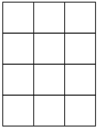
Formula for $\left((\chi^{(a^b)})^{\widetilde{\times 2}}\chi^{\nu}\right)\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$

For $a, b \in \mathbb{N}$ we say that a partition λ of size $2ab$ is (a, b)-birectangular if for all $1 \leq i \leq 2b$ the equality $\lambda_i + \lambda_{2b+1-i} = 2a$ holds true.

Proposition

The character $\left((\chi^{(a^b)})^{\widetilde{\times 2}}\chi^{\nu}\right)\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ with $\nu = (2)$, respectively, $\nu = (1^2)$ is multiplicity-free with constituents labelled by (a, b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even, respectively, odd.

Example: One of the irreducible constituents of $\left((\chi^{(3^2)})^{\widetilde{\times 2}}\chi^{(2)}\right)\uparrow_{S_6\wr S_2}^{S_{12}}$ is labelled by (3^4).



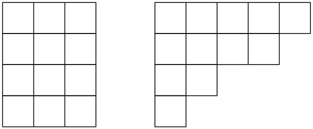
Formula for $\left((\chi^{(a^b)})^{\widetilde{\times 2}}\chi^{\nu}\right)\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$

For $a, b \in \mathbb{N}$ we say that a partition λ of size $2ab$ is (a, b)-birectangular if for all $1 \leq i \leq 2b$ the equality $\lambda_i + \lambda_{2b+1-i} = 2a$ holds true.

Proposition

The character $\left((\chi^{(a^b)})^{\widetilde{\times 2}}\chi^{\nu}\right)\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ with $\nu = (2)$, respectively, $\nu = (1^2)$ is multiplicity-free with constituents labelled by (a, b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even, respectively, odd.

Example: One of the irreducible constituents of $\left((\chi^{(3^2)})^{\widetilde{\times 2}}\chi^{(2)}\right)\uparrow_{S_6\wr S_2}^{S_{12}}$ is labelled by (3^4). If we consider $\left((\chi^{(3^2)})^{\widetilde{\times 2}}\chi^{(1^2)}\right)\uparrow_{S_6\wr S_2}^{S_{12}}$ instead, we obtain an irreducible constituent labelled by $(5, 4, 2, 1)$.



Birectangular partitions

Proposition (T, 2023)

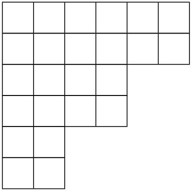
Let $a > b$ be positive integers and write $d = a - b$. There is a partition λ such that λ and λ' are (a, b)-birectangular if and only if $d|a$. Moreover, in such a case $\lambda = ((2b)^{2d}, (2b - 2d)^{2d}, \dots, (2d)^{2d})$.

Birectangular partitions

Proposition (T, 2023)

Let $a > b$ be positive integers and write $d = a - b$. There is a partition λ such that λ and λ' are (a, b)-birectangular if and only if $d | a$. Moreover, in such a case $\lambda = ((2b)^{2d}, (2b - 2d)^{2d}, \ldots, (2d)^{2d})$.

Example: The unique partition $\lambda = (6^2, 4^2, 2^2)$ such that λ and λ' are both $(4, 3)$-birectangular.



Examples

Recall the setting $G \neq N := G \cap A_n$ and the following results.

Examples

Recall the setting $G \neq N := G \cap A_n$ and the following results.

- If $\rho \neq \rho \times \text{sgn}$ is an irreducible induced-multiplicity-free character of G, then $\rho\big\downarrow_N^G$ is induced-multiplicity-free if and only if $\rho\big\uparrow_G^{S_n}$ does not have a pair of irreducible constituents χ^λ and $\chi^{\lambda'}$.

Examples

Recall the setting $G \neq N := G \cap A_n$ and the following results.

- If $\rho \neq \rho \times \mathrm{sgn}$ is an irreducible induced-multiplicity-free character of G, then $\rho\big\downarrow_N^G$ is induced-multiplicity-free if and only if $\rho\big\uparrow_G^{S_n}$ does not have a pair of irreducible constituents χ^λ and $\chi^{\lambda'}$.
- The character $\left((\chi^{(a^b)})^{\widetilde{\times 2}}\chi^\nu\right)\big\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ is multiplicity-free with irreducible constituents labelled by (a,b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even for $\nu = (2)$, respectively, odd for $\nu = (1^2)$.

Examples

Recall the setting $G \neq N := G \cap A_n$ and the following results.

- If $\rho \neq \rho \times \mathrm{sgn}$ is an irreducible induced-multiplicity-free character of G, then $\rho\big\downarrow_N^G$ is induced-multiplicity-free if and only if $\rho\big\uparrow_G^{S_n}$ does not have a pair of irreducible constituents χ^λ and $\chi^{\lambda'}$.
- The character $\left((\chi^{(a^b)})^{\widetilde{\times 2}}\chi^\nu\right)\big\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ is multiplicity-free with irreducible constituents labelled by (a,b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even for $\nu = (2)$, respectively, odd for $\nu = (1^2)$.
- There is λ with λ and λ' (a,b)-birectangular if and only if $a - b \mid a$. In such a case λ is even.

Examples

Recall the setting $G \neq N := G \cap A_n$ and the following results.

- If $\rho \neq \rho \times \mathrm{sgn}$ is an irreducible induced-multiplicity-free character of G, then $\rho\big\downarrow_N^G$ is induced-multiplicity-free if and only if $\rho\big\uparrow_G^{S_n}$ does not have a pair of irreducible constituents χ^λ and $\chi^{\lambda'}$.
- The character $\left((\chi^{(a^b)})^{\widetilde{\times 2}}\chi^\nu\right)\big\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ is multiplicity-free with irreducible constituents labelled by (a,b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even for $\nu = (2)$, respectively, odd for $\nu = (1^2)$.
- There is λ with λ and λ' (a,b)-birectangular if and only if $a - b \mid a$. In such a case λ is even.

Example: Let $G = S_{15} \wr S_2$. Then $\left((\chi^{(5^3)})^{\widetilde{\times 2}}\chi^{(2)}\right)\big\downarrow_N^G\big\uparrow_N^{S_{30}}$ is multiplicity-free.

Examples

Recall the setting $G \neq N := G \cap A_n$ and the following results.

- If $\rho \neq \rho \times \text{sgn}$ is an irreducible induced-multiplicity-free character of G, then $\rho\downarrow_N^G$ is induced-multiplicity-free if and only if $\rho\uparrow_G^{S_n}$ does not have a pair of irreducible constituents χ^λ and $\chi^{\lambda'}$.
- The character $\left((\chi^{(a^b)})^{\widetilde{\times 2}}\chi^\nu\right)\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ is multiplicity-free with irreducible constituents labelled by (a,b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even for $\nu = (2)$, respectively, odd for $\nu = (1^2)$.
- There is λ with λ and λ' (a,b)-birectangular if and only if $a - b \mid a$. In such a case λ is even.

Example: Let $G = S_{15} \wr S_2$. Then $\left((\chi^{(5^3)})^{\widetilde{\times 2}}\chi^{(1^2)}\right)\downarrow_N^G\uparrow_N^{S_{30}}$ is multiplicity-free.

Examples

Recall the setting $G \neq N := G \cap A_n$ and the following results.

- If $\rho \neq \rho \times \mathrm{sgn}$ is an irreducible induced-multiplicity-free character of G, then $\rho\big\downarrow_N^G$ is induced-multiplicity-free if and only if $\rho\big\uparrow_G^{S_n}$ does not have a pair of irreducible constituents χ^λ and $\chi^{\lambda'}$.
- The character $\left((\chi^{(a^b)})^{\widetilde{\times 2}}\chi^\nu\right)\big\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ is multiplicity-free with irreducible constituents labelled by (a,b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even for $\nu = (2)$, respectively, odd for $\nu = (1^2)$.
- There is λ with λ and λ' (a,b)-birectangular if and only if $a - b \mid a$. In such a case λ is even.

Example: Let $G = S_{12} \wr S_2$. Then $\left((\chi^{(4^3)})^{\widetilde{\times 2}}\chi^{((1^2))}\right)\big\downarrow_N^G\big\uparrow_N^{S_{24}}$ is multiplicity-free.

Examples

Recall the setting $G \neq N := G \cap A_n$ and the following results.

- If $\rho \neq \rho \times \mathrm{sgn}$ is an irreducible induced-multiplicity-free character of G, then $\rho\big\downarrow_N^G$ is induced-multiplicity-free if and only if $\rho\big\uparrow_G^{S_n}$ does not have a pair of irreducible constituents χ^λ and $\chi^{\lambda'}$.
- The character $\left((\chi^{(a^b)})^{\widetilde{\times 2}}\chi^\nu\right)\big\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ is multiplicity-free with irreducible constituents labelled by (a,b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even for $\nu = (2)$, respectively, odd for $\nu = (1^2)$.
- There is λ with λ and λ' (a,b)-birectangular if and only if $a - b \mid a$. In such a case λ is even.

Example: Let $G = S_{12} \wr S_2$. Then $\left((\chi^{(4^3)})^{\widetilde{\times 2}}\chi^{(2)}\right)\big\downarrow_N^G\big\uparrow_N^{S_{24}}$ is not multiplicity-free.

Outline

Multiplicity-free subgroups of S_n

Theorem (T, 2023)

Let $n \geq 66$. A subgroup $G \leq S_n$ is multiplicity-free if and only if it belongs to the list (throughout $k, l \geq 1$, $m \geq 2$ and $h \geq 3$):

1. S_n and A_n,
2. $S_k \times S_l$, $(S_k \times S_l) \cap A_{k+l}$ and $A_k \times S_l$ with $k \neq 2$ and $A_k \times A_l$ with $k, l \neq 2$,
3. $S_k \times S_m \wr S_2$ and for $k \notin \{2m-3, 2m-2, 2m-1, 2m\}$ also groups $(S_k \times S_m \wr S_2) \cap A_{k+2m}$ and $T_{k,m,2}$,
4. $S_k \times S_2 \wr S_h$ and for $k \geq 2h+2$ also $(S_k \times S_2 \wr S_h) \cap A_{k+2h}$,
5. $S_m \wr S_2, (S_m \wr S_2) \cap A_{2m}, A_m \wr S_2$ and $T_{m,2}$,
6. $S_2 \wr S_h$ and $(S_2 \wr S_h) \cap A_{2h}$,
7. $S_m \wr S_3, (S_m \wr S_3) \cap A_{3m}$ and $T_{m,3}$,
8. $S_k \times L, A_k \times L$ and $(S_k \times L) \cap A_n$ where L is one of $\mathrm{P\Gamma L}(2,8), \mathrm{ASL}(3,2)$, $\mathrm{PGL}(2,5)$ and $\mathrm{AGL}(1,5)$,
9. $A_k \times S_2 \wr S_2, N_k$ and $T_{k,2,h}$ with $h \in \{3,4\}$,
10. $A_m \wr S_2$ with $n = 2m+1$ provided m is a square.

Multiplicity-free subgroups of S_n

Theorem (T, 2023)

Let $n \geq 66$. A subgroup $G \leq S_n$ is multiplicity-free if and only if it belongs to the list (throughout $k, l \geq 1$, $m \geq 2$ and $h \geq 3$):

1. S_n and A_n,
2. $S_k \times S_l$, $(S_k \times S_l) \cap A_{k+l}$ and $A_k \times S_l$ with $k \neq 2$ and $A_k \times A_l$ with $k, l \neq 2$,
3. $S_k \times S_m \wr S_2$ and for $k \notin \{2m-3, 2m-2, 2m-1, 2m\}$ also groups $(S_k \times S_m \wr S_2) \cap A_{k+2m}$ and $T_{k,m,2}$,
4. $S_k \times S_2 \wr S_h$ and for $k \geq 2h+2$ also $(S_k \times S_2 \wr S_h) \cap A_{k+2h}$,
5. $S_m \wr S_2, (S_m \wr S_2) \cap A_{2m}, A_m \wr S_2$ and $T_{m,2}$,
6. $S_2 \wr S_h$ and $(S_2 \wr S_h) \cap A_{2h}$,
7. $S_m \wr S_3, (S_m \wr S_3) \cap A_{3m}$ and $T_{m,3}$,
8. $S_k \times L, A_k \times L$ and $(S_k \times L) \cap A_n$ where L is one of $\mathrm{P\Gamma L}(2,8), \mathrm{ASL}(3,2)$, $\mathrm{PGL}(2,5)$ and $\mathrm{AGL}(1,5)$,
9. $A_k \times S_2 \wr S_2, N_k$ and $T_{k,2,h}$ with $h \in \{3,4\}$,
10. $A_m \wr S_2$ with $n = 2m+1$ provided m is a square.

Induced-multiplicity-free characters of $S_k \times S_2 \wr S_h$

To demonstrate our partial answer to **Q2** we present the irreducible induced-multiplicity-free characters of $S_k \times S_2 \wr S_h$ (with $k \geq 2$ and $h \geq 5$) and $(S_m \wr S_2) \cap A_{2m}$.

Induced-multiplicity-free characters of $S_k \times S_2 \wr S_h$

To demonstrate our partial answer to **Q2** we present the irreducible induced-multiplicity-free characters of $S_k \times S_2 \wr S_h$ (with $k \geq 2$ and $h \geq 5$) and $(S_m \wr S_2) \cap A_{2m}$.

Theorem (T, 2023)

Let $k \geq 2, h \geq 5$ with $k + 2h \geq 18$. The irreducible induced-multiplicity-free characters of $G = S_k \times S_2 \wr S_h$ are $\chi^{(1^k)} \boxtimes \left(\widetilde{(\chi^{(2)})^{\times h}} \chi^{(h)} \right)$ and $\chi^{(k)} \boxtimes \left(\widetilde{(\chi^{(1^2)})^{\times h}} \chi^{(h)} \right)$.

Induced-multiplicity-free characters of $S_k \times S_2 \wr S_h$

To demonstrate our partial answer to **Q2** we present the irreducible induced-multiplicity-free characters of $S_k \times S_2 \wr S_h$ (with $k \geq 2$ and $h \geq 5$) and $(S_m \wr S_2) \cap A_{2m}$.

Theorem (T, 2023)

Let $k \geq 2, h \geq 5$ with $k + 2h \geq 18$. The irreducible induced-multiplicity-free characters of $G = S_k \times S_2 \wr S_h$ are $\chi^{(1^k)} \boxtimes \left(\widetilde{(\chi^{(2)})^{\times h}} \chi^{(h)} \right)$ and $\chi^{(k)} \boxtimes \left(\widetilde{(\chi^{(1^2)})^{\times h}} \chi^{(h)} \right)$.

In the language of symmetric functions $s_\lambda(s_\nu \circ s_\mu)$ with $|\lambda| \geq 2 = |\mu|$ and $|\nu| \geq 5$ is multiplicity-free if and only if ν is a row partition and λ and μ are in some order a row and a column partition.

Induced-multiplicity-free characters of $S_k \times S_2 \wr S_h$

To demonstrate our partial answer to **Q2** we present the irreducible induced-multiplicity-free characters of $S_k \times S_2 \wr S_h$ (with $k \geq 2$ and $h \geq 5$) and $(S_m \wr S_2) \cap A_{2m}$.

Theorem (T, 2023)

Let $k \geq 2, h \geq 5$ with $k + 2h \geq 18$. The irreducible induced-multiplicity-free characters of $G = S_k \times S_2 \wr S_h$ are $\chi^{(1^k)} \boxtimes \left(\widetilde{(\chi^{(2)})^{\times h}} \chi^{(h)} \right)$ and $\chi^{(k)} \boxtimes \left(\widetilde{(\chi^{(1^2)})^{\times h}} \chi^{(h)} \right)$.

In the language of symmetric functions $s_\lambda(s_\nu \circ s_\mu)$ with $|\lambda| \geq 2 = |\mu|$ and $|\nu| \geq 5$ is multiplicity-free if and only if ν is a row partition and λ and μ are in some order a row and a column partition. Observe that $S_k \times S_2 \wr S_h$ (with the above constraints) is multiplicity-free but its trivial character is not induced-multiplicity-free.

Induced-multiplicity-free characters of $(S_m \wr S_2) \cap A_{2m}$

Theorem (T, 2023)

Let $m \geq 37$. The elementary irreducible induced-multiplicity-free characters of $G = (S_m \wr S_2) \cap A_{2m}$ are the irreducible constituents of $\left((\chi^\mu)^{\widetilde{\times 2}} \chi^\nu\right) \downarrow_G^{S_m \wr S_2}$ with:

1. μ of the form (a^b) such that $a - b \nmid a$ or $\nu = (1^2)$;
2. μ of the form $(a+1, b)$ with $b > a+1$;
3. μ of the form $(a+1, a^{b-1}), (a^b, 1)$ or $(a^{b-1}, a-1)$ with $a > b+2$ and $a - b \nmid a$;
4. μ of the form $((2b)^{b-1}, 2b-1), (2b+1, (2b)^{b-1})$ or $(3b+1, (3b)^{2b-1})$;
5. μ of the form $(a^{a-1}, a-1)$ provided m is even.

Induced-multiplicity-free characters of $(S_m \wr S_2) \cap A_{2m}$

Theorem (T, 2023)

Let $m \geq 37$. The elementary irreducible induced-multiplicity-free characters of $G = (S_m \wr S_2) \cap A_{2m}$ are the irreducible constituents of $\left((\chi^\mu)^{\widetilde{\times 2}} \chi^\nu\right) \downarrow_G^{S_m \wr S_2}$ with:

1. μ of the form (a^b) such that $a - b \nmid a$ or $\nu = (1^2)$;
2. μ of the form $(a+1, b)$ with $b > a+1$;
3. μ of the form $(a+1, a^{b-1}), (a^b, 1)$ or $(a^{b-1}, a-1)$ with $a > b+2$ and $a - b \nmid a$;
4. μ of the form $((2b)^{b-1}, 2b-1), (2b+1, (2b)^{b-1})$ or $(3b+1, (3b)^{2b-1})$;
5. μ of the form $(a^{a-1}, a-1)$ provided m is even.

Induced-multiplicity-free characters of $(S_m \wr S_2) \cap A_{2m}$

Theorem (T, 2023)

Let $m \geq 37$. The elementary irreducible induced-multiplicity-free characters of $G = (S_m \wr S_2) \cap A_{2m}$ are the irreducible constituents of $\left((\chi^\mu)^{\widetilde{\times 2}} \chi^\nu\right) \downarrow_G^{S_m \wr S_2}$ with:

1. μ of the form (a^b) such that $a - b \nmid a$ or $\nu = (1^2)$;
2. μ of the form $(a+1, b)$ with $b > a+1$;
3. μ of the form $(a+1, a^{b-1}), (a^b, 1)$ or $(a^{b-1}, a-1)$ with $a > b+2$ and $a - b \nmid a$;
4. μ of the form $((2b)^{b-1}, 2b-1), (2b+1, (2b)^{b-1})$ or $(3b+1, (3b)^{2b-1})$;
5. μ of the form $(a^{a-1}, a-1)$ provided m is even.

Induced-multiplicity-free characters of $(S_m \wr S_2) \cap A_{2m}$

Theorem (T, 2023)

Let $m \geq 37$. The elementary irreducible induced-multiplicity-free characters of $G = (S_m \wr S_2) \cap A_{2m}$ are the irreducible constituents of $\left((\chi^\mu)^{\widetilde{\times 2}} \chi^\nu\right) \downarrow_G^{S_m \wr S_2}$ with:

1. μ of the form (a^b) such that $a - b \nmid a$ or $\nu = (1^2)$;
2. μ of the form $(a+1, b)$ with $b > a+1$;
3. μ of the form $(a+1, a^{b-1}), (a^b, 1)$ or $(a^{b-1}, a-1)$ with $a > b+2$ and $a - b \nmid a$;
4. μ of the form $((2b)^{b-1}, 2b-1), (2b+1, (2b)^{b-1})$ or $(3b+1, (3b)^{2b-1})$;
5. μ of the form $(a^{a-1}, a-1)$ provided m is even.