Multiplicity-free induced characters of symmetric groups

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October 11, 2023

Based on arXiv:2309.07761

Outline

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1. Introduction
2. Representation theory of symmetric groups in characteristic 0
3. Necessary conditions for multiplicity-free subgroups
4. Combinatorics for index two subgroups
5. Main results

Multiplicity-free characters

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Q1: What are the multiplicity-free subgroups of S_n?
Q2: For each multiplicity-free subgroup $G \leq S_n$ what are the irreducible induced-multiplicity-free characters of G?

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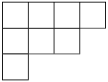
Example: The Young diagram of the partition $(4, 3, 1)$.



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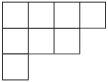

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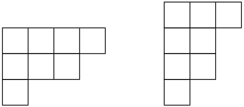

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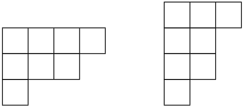

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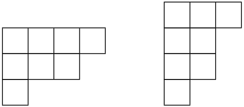

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As a consequence of our work a classification of multiplicity-free symmetric functions $s_\lambda(s_\nu \circ s_\mu)$ is found. Moreover, for n sufficiently large, one can observe this is the only way to 'compose' 3 or more Schur functions and get a multiplicity-free symmetric function.

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The lower bound on $|G|$ also eliminates, for example, the diagonal subgroup of $S_k \times S_k$.

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These results impose restrictions on non-elementary irreducible induced-multiplicity-free characters of wreath products.

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Example: Let $G = S_2 \times S_3$. Since $\left(\chi^{(2)} \boxtimes \chi^{(2,1)}\right)\big\uparrow_G^{S_5}$ decomposes as $\chi^{(4,1)} + \chi^{(3,2)} + \chi^{(3,1^2)} + \chi^{(2^2,1)}$, the character $\left(\chi^{(2)} \boxtimes \chi^{(2,1)}\right)\big\downarrow_N^G$ is not induced-multiplicity-free.

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Lemma

Let h be a non-negative integer. There is an even partition λ of $2h$ such that λ' is also even if and only if h is even.

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Example: Let $G = S_2 \wr S_{11}$. Then $(\chi^{(2)})^{\widetilde{\times 11}}\chi^{11}\downarrow_N^G$ is induced-multiplicity-free.

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Example: Let $G = S_2 \wr S_{10}$. Then $(\chi^{(2)})^{\widetilde{\times 10}}\chi^{10}\downarrow_N^G$ is not induced-multiplicity-free.

Formula for $\left((\chi^{(a^b)})^{\widetilde{\times 2}}\chi^{\nu}\right)\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$

For $a, b \in \mathbb{N}$ we say that a partition λ of size $2ab$ is (a, b)-birectangular if for all $1 \leq i \leq 2b$ the equality $\lambda_i + \lambda_{2b+1-i} = 2a$ holds true.

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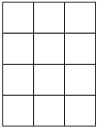
Example: One of the irreducible constituents of $\left((\chi^{(3^2)})^{\widetilde{\times 2}}\chi^{(2)}\right)\uparrow_{S_6\wr S_2}^{S_{12}}$ is labelled by (3^4).



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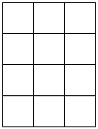
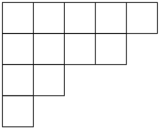

Birectangular partitions I

Proposition (T, 2023)

Let $a > b$ be positive integers and write $d = a - b$. There is a partition λ such that λ and λ' are (a, b)-birectangular if and only if $d|a$. Moreover, in such a case $\lambda = ((2b)^{2d}, (2b - 2d)^{2d}, \dots, (2d)^{2d})$.

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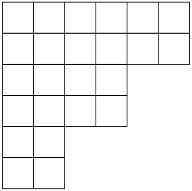
Example: The unique partition $\lambda = (6^2, 4^2, 2^2)$ such that λ and λ' are both $(4, 3)$-birectangular.



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Proposition (T, 2023)

Let t be a non-negative integer. Suppose that $a > b$ are positive integers. Write $d = a - b$ and $0 \leq r \leq d - 1$ for the remainder of a modulo d. There is a partition λ of $2ab + t$ such that for some (a, b)-birectangular partitions μ and ν we have $\mu, \nu' \subseteq \lambda$ if and only if $t \geq 2r(d - r)$.

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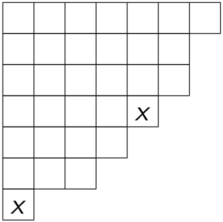
Example: $(7, 6^2, 5, 4, 3, 1)$ has size 32 and contains a $(5, 3)$-birectangular partition.



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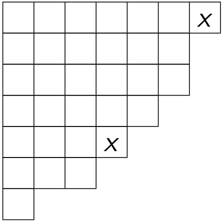

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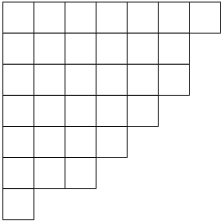

Outline

Multiplicity-free subgroups of S_n (light version)

Theorem (T, 2023)

Suppose that $n \geq 66$. Multiplicity-free subgroups of S_n are given by certain subgroups of index $1, 2$ and 4 of

1. $S_k \times S_l$,
2. $S_k \times S_m \wr S_2$,
3. $S_k \times S_2 \wr S_h$,
4. $S_m \wr S_3$,
5. $S_k \times L$ where L is one of $\mathrm{P\Gamma L}_2(\mathbb{F}_8) \leq S_9$, $\mathrm{ASL}_3(\mathbb{F}_2) \leq S_8$, $\mathrm{PGL}_2(\mathbb{F}_5) \leq S_6$ and $\mathrm{AGL}_1(\mathbb{F}_5) \leq S_5$,

where $k, l \geq 0$, $m \geq 2$ and $h \geq 3$.

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where $k, l \geq 0$, $m \geq 2$ and $h \geq 3$.

Note that the trivial characters of groups in (2), (3) and (4) are not induced-multiplicity-free, provided $k \geq 2$.

Multiplicity-free subgroups of S_n (full version)

Theorem (T, 2023)

Let $n \geq 66$. A subgroup $G \leq S_n$ is multiplicity-free if and only if it belongs to the list (throughout $k, l \geq 1$, $m \geq 2$ and $h \geq 3$):

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2. $S_k \times S_l$, $(S_k \times S_l) \cap A_{k+l}$ and $A_k \times S_l$ with $k \neq 2$ and $A_k \times A_l$ with $k, l \neq 2$,
3. $S_k \times S_m \wr S_2$ and for $k \notin \{2m-3, 2m-2, 2m-1, 2m\}$ also groups $(S_k \times S_m \wr S_2) \cap A_{k+2m}$ and $T_{k,m,2}$,
4. $S_k \times S_2 \wr S_h$ and for $k \geq 2h+2$ also $(S_k \times S_2 \wr S_h) \cap A_{k+2h}$,
5. $S_m \wr S_2, (S_m \wr S_2) \cap A_{2m}, A_m \wr S_2$ and $T_{m,2}$,
6. $S_2 \wr S_h$ and $(S_2 \wr S_h) \cap A_{2h}$,
7. $S_m \wr S_3, (S_m \wr S_3) \cap A_{3m}$ and $T_{m,3}$,
8. $S_k \times L, A_k \times L$ and $(S_k \times L) \cap A_n$ where L is one of $\mathrm{P\Gamma L}(2,8), \mathrm{ASL}(3,2)$, $\mathrm{PGL}(2,5)$ and $\mathrm{AGL}(1,5)$,
9. $A_k \times S_2 \wr S_2, N_k$ and $T_{k,2,h}$ with $h \in \{3,4\}$,
10. $A_m \wr S_2$ with $n = 2m+1$ provided m is a square.

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4. $S_k \times S_2 \wr S_h$ and for $k \geq 2h+2$ also $(S_k \times S_2 \wr S_h) \cap A_{k+2h}$,
5. $S_m \wr S_2, (S_m \wr S_2) \cap A_{2m}, A_m \wr S_2$ and $T_{m,2}$,
6. $S_2 \wr S_h$ and $(S_2 \wr S_h) \cap A_{2h}$,
7. $S_m \wr S_3, (S_m \wr S_3) \cap A_{3m}$ and $T_{m,3}$,
8. $S_k \times L, A_k \times L$ and $(S_k \times L) \cap A_n$ where L is one of $\mathrm{P\Gamma L}(2,8), \mathrm{ASL}(3,2), \mathrm{PGL}(2,5)$ and $\mathrm{AGL}(1,5)$,
9. $A_k \times S_2 \wr S_2, N_k$ and $T_{k,2,h}$ with $h \in \{3,4\}$,
10. $A_m \wr S_2$ with $n = 2m+1$ provided m is a square.

Induced-multiplicity-free characters of $(S_2 \wr S_h) \cap A_{2h}$

To demonstrate our partial answer to **Q2** we present the irreducible induced-multiplicity-free characters of $(S_2 \wr S_h) \cap A_{2h}$ and $(S_m \wr S_2) \cap A_{2m}$.

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Let $h \geq 33$. The irreducible induced-multiplicity-free characters of $G = (S_2 \wr S_h) \cap A_{2h}$ are $\left((\chi^{(2)})^{\widetilde{\times h}}\chi^{(1^h)}\right) \downarrow^{S_2 \wr S_h}_G$ and for odd h also $\left((\chi^{(2)})^{\widetilde{\times h}}\chi^{(h)}\right) \downarrow^{S_2 \wr S_h}_G$.

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The second character appears only for odd h as noted in an earlier example.

Induced-multiplicity-free characters of $(S_m \wr S_2) \cap A_{2m}$

Theorem (T, 2023)

Let $m \geq 37$. The elementary irreducible induced-multiplicity-free characters of $G = (S_m \wr S_2) \cap A_{2m}$ are the irreducible constituents of $\left((\chi^\mu)^{\widetilde{\times 2}} \chi^\nu\right) \downarrow_G^{S_m \wr S_2}$ with:

1. μ of the form (a^b) such that $a - b \nmid a$ or $\nu = (1^2)$;
2. μ of the form $(a+1, b)$ with $b > a+1$;
3. μ of the form $(a+1, a^{b-1}), (a^b, 1)$ or $(a^{b-1}, a-1)$ with $a > b+2$ and $a - b \nmid a$;
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