Multiplicity-free induced characters of symmetric groups

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Example: Any irreducible character is multiplicity-free. The character of the regular representation of G is multiplicity-free if and only if G is abelian.

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Q1: What are the multiplicity-free subgroups of S_n?
Q2: For each multiplicity-free subgroup $G \leq S_n$ what are the irreducible induced-multiplicity-free characters of G?

Multiplicity-free characters of S_n in literature

The classification of multiplicity-free permutation characters of S_n - Wildon in 2009 (for $n \geq 66$) and independently Godsil and Meagher in 2010.

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Our **Q1** asks for 'weak Gelfand pairs' (S_n, G).

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Theorem (Maróti, 2002)

Let G be a primitive subgroup of S_n which is not S_n or A_n. Then the order of G is less or equal to 2^{n-1} or $n \leq 24$.

Non-transitive subgroups

Multiplicity-free subgroups have at most two orbits as shown by Stembridge.

Theorem (Stembridge, 2001)

Let $n = a + b + c$ for positive integers a, b and c. Then $S_a \times S_b \times S_c$ is not multiplicity-free.

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This impacts irreducible induced-multiplicity-free characters of wreath products.

Partitions

A **partition of** n is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ of positive integers with $\sum_{i=1}^{t} \lambda_i = n$.

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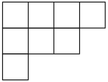
Example: The Young diagram of the partition $(4, 3, 1)$.



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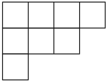
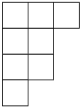


Its **conjugate partition** is $(3, 2, 2, 1)$.



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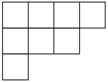
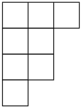


Its **conjugate partition** is $(3, 2, 2, 1)$ and is often denoted as $(3, 2^2, 1)$.



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The irreducible characters of $S_k \times S_l$ are $\chi^\mu \boxtimes \chi^\nu$ labelled by partitions μ and ν of k and l, respectively.
The irreducible characters of $S_m \wr S_h$ are constructed from the **elementary irreducible characters** $\chi^\mu \wr \chi^\nu := \widetilde{(\chi^\mu)^{\times h}}\chi^\nu$ labelled by partitions μ and ν of m and h, respectively.

Index two subgroups

Suppose that $G \neq N := G \cap A_n$ and ρ is an irreducible character of G such that $\rho \times \mathrm{sgn} \neq \rho$.

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Example: Let $G = S_2 \times S_3$ and $\rho = \chi^{(2)} \boxtimes \chi^{(2,1)}$. Since $\rho\big\uparrow_G^{S_5}$ decomposes as $\chi^{(4,1)} + \chi^{(3,2)} + \chi^{(3,1^2)} + \chi^{(2^2,1)}$, the character $\rho\big\downarrow_N^G$ is not induced-multiplicity-free.

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Formula for $\left(\chi^{(a^b)} \wr \chi^{(1^2)}\right) \uparrow_{S_{ab} \wr S_2}^{S_{2ab}}$

For $a, b \in \mathbb{N}$ we say that a partition λ of size $2ab$ is (a, b)-**birectangular** if for all $1 \leq i \leq 2b$ the equality $\lambda_i + \lambda_{2b+1-i} = 2a$ holds true.

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Let $\rho = \chi^{(a^b)} \wr \chi^{(1^2)}$. The character $\rho\uparrow_{S_{ab} \wr S_2}^{S_{2ab}}$ is multiplicity-free with constituents labelled by (a, b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is odd.

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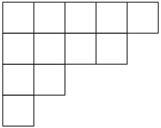
Example: Let $\rho = \chi^{(3^2)} \wr \chi^{(1^2)}$. One of the irreducible constituents of $\rho\big\uparrow_{S_6 \wr S_2}^{S_{12}}$ is labelled by $(5, 4, 2, 1)$.



Birectangular partitions

Proposition (T, 2023)

Let $a > b$ be positive integers and write $d = a - b$. There is a partition λ such that λ and λ' are (a, b)-birectangular if and only if $d|a$. Moreover, in such a case $\lambda = ((2b)^{2d}, (2b - 2d)^{2d}, \dots, (2d)^{2d})$.

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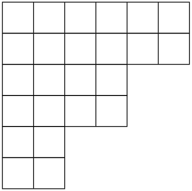
Example: The unique partition $\lambda = (6^2, 4^2, 2^2)$ such that λ and λ' are both $(4, 3)$-birectangular.



Multiplicity-free subgroups of S_n (light version)

Theorem (T, 2023)

Suppose that $n \geq 66$. Multiplicity-free subgroups of S_n are given by certain subgroups of index $1, 2$ and 4 of

1. $S_k \times S_l$,
2. $S_k \times S_m \wr S_2$,
3. $S_k \times S_2 \wr S_h$,
4. $S_m \wr S_3$,
5. $S_k \times L$ where L is one of $\mathrm{P\Gamma L}_2(\mathbb{F}_8) \leq S_9$, $\mathrm{ASL}_3(\mathbb{F}_2) \leq S_8$, $\mathrm{PGL}_2(\mathbb{F}_5) \leq S_6$ and $\mathrm{AGL}_1(\mathbb{F}_5) \leq S_5$,

where $k, l \geq 0$, $m \geq 2$ and $h \geq 3$.

Multiplicity-free subgroups of S_n (full version)

Theorem (T, 2023)

Let $n \geq 66$. A subgroup $G \leq S_n$ is multiplicity-free if and only if it belongs to the list (throughout $k, l \geq 1$, $m \geq 2$ and $h \geq 3$):

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3. $S_k \times S_m \wr S_2$ and for $k \notin \{2m-3, 2m-2, 2m-1, 2m\}$ also groups $(S_k \times S_m \wr S_2) \cap A_{k+2m}$ and $T_{k,m,2}$,
4. $S_k \times S_2 \wr S_h$ and for $k \geq 2h+2$ also $(S_k \times S_2 \wr S_h) \cap A_{k+2h}$,
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7. $S_m \wr S_3, (S_m \wr S_3) \cap A_{3m}$ and $T_{m,3}$,
8. $S_k \times L, A_k \times L$ and $(S_k \times L) \cap A_n$ where L is one of $\mathrm{P\Gamma L}(2,8), \mathrm{ASL}(3,2)$, $\mathrm{PGL}(2,5)$ and $\mathrm{AGL}(1,5)$,
9. $A_k \times S_2 \wr S_2, N_k$ and $T_{k,2,h}$ with $h \in \{3,4\}$,
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Induced-multiplicity-free characters of $(S_m \wr S_2) \cap A_{2m}$

Theorem (T, 2023)

Let $m \geq 37$. The elementary irreducible induced-multiplicity-free characters of $G = (S_m \wr S_2) \cap A_{2m}$ are the irreducible constituents of $(\chi^\mu \wr \chi^\nu) \downarrow_G^{S_m \wr S_2}$ with:

1. μ of the form (a^b) such that $a - b \nmid a$ or $\nu = (1^2)$;
2. μ of the form $(a+1, b)$ with $b > a+1$;
3. μ of the form $(a+1, a^{b-1}), (a^b, 1)$ or $(a^{b-1}, a-1)$ with $a > b+2$ and $a - b \nmid a$;
4. μ of the form $((2b)^{b-1}, 2b-1), (2b+1, (2b)^{b-1})$ or $(3b+1, (3b)^{2b-1})$;
5. μ of the form $(a^{a-1}, a-1)$ provided m is even.

Induced-multiplicity-free characters of $(S_m \wr S_2) \cap A_{2m}$

Theorem (T, 2023)

Let $m \geq 37$. The elementary irreducible induced-multiplicity-free characters of $G = (S_m \wr S_2) \cap A_{2m}$ are the irreducible constituents of $(\chi^\mu \wr \chi^\nu) \downarrow_G^{S_m \wr S_2}$ with:

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