

Multiplicity-free induced characters of symmetric groups

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Outline

- 1 Introduction
- 2 Representation theory of symmetric groups in characteristic 0
- 3 Necessary conditions for multiplicity-free subgroups
- 4 Combinatorics for index two subgroups
- 5 Main results

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Example: Any irreducible character is multiplicity-free. The character of the regular representation of G is multiplicity-free if and only if G is abelian.

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Q1: What are the multiplicity-free subgroups of S_n ?

Q2: For each multiplicity-free subgroup $G \leq S_n$ what are the irreducible induced-multiplicity-free characters of G ?

Multiplicity-free characters of S_n in literature

The classification of multiplicity-free permutation characters of S_n - Wildon in 2009 (for $n \geq 66$) and independently Godsil and Meagher in 2010.

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Our **Q1** asks for 'weak Gelfand pairs' (S_n, G) .

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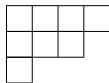
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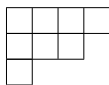


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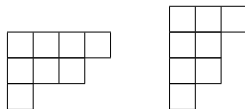
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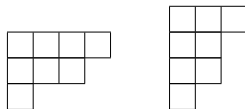
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In fact, for n sufficiently large, these are the only multiplicity-free 'compositions' of 3 or more Schur functions.

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The lower bound on $|G|$ also eliminates, for example, the diagonal subgroup of $S_k \times S_k$.

Non-transitive subgroups

Multiplicity-free subgroups have at most two orbits as shown by Stembridge.

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This impacts non-elementary irreducible induced-multiplicity-free characters of wreath products.

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For $a, b \in \mathbb{N}$ we say that a partition λ of size $2ab$ is (a, b) -**birectangular** if for all $1 \leq i \leq 2b$ the equality $\lambda_i + \lambda_{2b+1-i} = 2a$ holds true.

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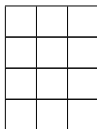
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Example: Let $\rho = \chi^{(3^2)} \wr \chi^{(2)}$. One of the irreducible constituents of $\rho \uparrow_{S_6 \wr S_2}^{S_{12}}$ is labelled by (3^4) .



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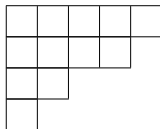
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Birectangular partitions I

Proposition (T, 2024)

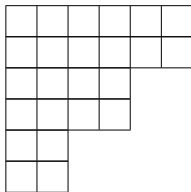
Let $a > b$ be positive integers and write $d = a - b$. There is a partition λ such that λ and λ' are (a, b) -birectangular if and only if $d|a$. Moreover, in such a case $\lambda = ((2b)^{2d}, (2b - 2d)^{2d}, \dots, (2d)^{2d})$.

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Example: The unique partition $\lambda = (6^2, 4^2, 2^2)$ such that λ and λ' are both $(4, 3)$ -birectangular.



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Birectangular partitions II

Proposition (T, 2024)

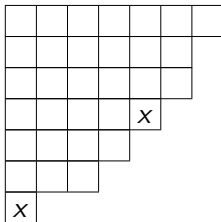
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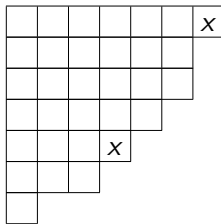


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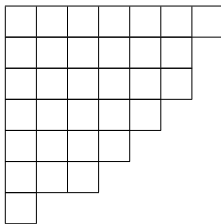


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Outline

- 1 Introduction
- 2 Representation theory of symmetric groups in characteristic 0
- 3 Necessary conditions for multiplicity-free subgroups
- 4 Combinatorics for index two subgroups
- 5 Main results**

Multiplicity-free subgroups of S_n (light version)

Theorem (T, 2024)

Suppose that $n \geq 66$. Multiplicity-free subgroups of S_n are given by certain subgroups of index 1, 2 and 4 of

- ① $S_k \times S_l$,
- ② $S_k \times S_m \wr S_2$,
- ③ $S_k \times S_2 \wr S_h$,
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The trivial characters of groups in (2), (3) and (4) are not induced-multiplicity-free, provided $k \geq 2$.

Multiplicity-free subgroups of S_n (full version)

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Let $n \geq 66$. A subgroup $G \leq S_n$ is multiplicity-free if and only if it belongs to the list (throughout $k, l \geq 1$, $m \geq 2$ and $h \geq 3$):

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To demonstrate the found answers to **Q2** we present the irreducible induced-multiplicity-free characters of $(S_2 \wr S_h) \cap A_{2h}$ and $(S_m \wr S_2) \cap A_{2m}$.

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Let $h \geq 33$. The irreducible induced-multiplicity-free characters of $G = (S_2 \wr S_h) \cap A_{2h}$ are $(\chi^{(2)} \wr \chi^{(1^h)}) \downarrow_G^{S_2 \wr S_h}$ and for odd h also $(\chi^{(2)} \wr \chi^{(h)}) \downarrow_G^{S_2 \wr S_h}$.

Induced-multiplicity-free characters of $(S_2 \wr S_h) \cap A_{2h}$

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The second character appears only for odd h as noted in an earlier example.

Induced-multiplicity-free characters of $(S_m \wr S_2) \cap A_{2m}$

Theorem (T, 2024)

Let $m \geq 37$. The elementary irreducible induced-multiplicity-free characters of $G = (S_m \wr S_2) \cap A_{2m}$ are the irreducible constituents of $(\chi^\mu \wr \chi^\nu) \downarrow_G^{S_m \wr S_2}$ with:

- 1 μ of the form (a^b) such that $a - b \nmid a$ or $\nu = (1^2)$;
- 2 μ of the form $(a + 1, b)$ with $b > a + 1$;
- 3 μ of the form $(a + 1, a^{b-1}), (a^b, 1)$ or $(a^{b-1}, a - 1)$ with $a > b + 2$ and $a - b \nmid a$;
- 4 μ of the form $((2b)^{b-1}, 2b - 1), (2b + 1, (2b)^{b-1})$ or $(3b + 1, (3b)^{2b-1})$;
- 5 μ of the form $(a^{a-1}, a - 1)$ provided m is even.

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- ① μ of the form (a^b) such that $a - b \nmid a$ or $\nu = (1^2)$;
- ② μ of the form $(a + 1, b)$ with $b > a + 1$;
- ③ μ of the form $(a + 1, a^{b-1}), (a^b, 1)$ or $(a^{b-1}, a - 1)$ with $a > b + 2$ and $a - b \nmid a$;
- ④ μ of the form $((2b)^{b-1}, 2b - 1), (2b + 1, (2b)^{b-1})$ or $(3b + 1, (3b)^{2b-1})$;
- ⑤ μ of the form $(a^{a-1}, a - 1)$ provided m is even.

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