Multiplicity-free induced characters of symmetric groups

Pavel Turek

Royal Holloway, University of London

April 24, 2023

Characters of symmetric groups

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ with *size* $\sum_{i=1}^{t} \lambda_i$.

Characters of symmetric groups

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ with *size* $\sum_{i=1}^{t} \lambda_i$. The *Young diagram* of λ is $Y_\lambda = \left\{(i,j) \in \mathbb{N}^2 : i \leq \ell(\lambda), j \leq \lambda_i\right\}$.

Characters of symmetric groups

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)$ with *size* $\sum_{i=1}^{t} \lambda_i$. The *Young diagram* of λ is $Y_\lambda = \left\{(i,j) \in \mathbb{N}^2 : i \leq \ell(\lambda), j \leq \lambda_i\right\}$. The *conjugate partition* of λ, denoted by λ', is a partition such that $Y_{\lambda'} = \{(i,j) : (j,i) \in Y_\lambda\}$.

Characters of symmetric groups

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ with *size* $\sum_{i=1}^{t} \lambda_i$. The *Young diagram* of λ is $Y_\lambda = \left\{(i,j) \in \mathbb{N}^2 : i \leq \ell(\lambda), j \leq \lambda_i\right\}$. The *conjugate partition* of λ, denoted by λ', is a partition such that $Y_{\lambda'} = \{(i,j) : (j,i) \in Y_\lambda\}$. Finally, we write $\mu \subseteq \lambda$ if $Y_\mu \subseteq Y_\lambda$.

Characters of symmetric groups

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ with *size* $\sum_{i=1}^{t} \lambda_i$. The *Young diagram* of λ is $Y_\lambda = \{(i,j) \in \mathbb{N}^2 : i \leq \ell(\lambda), j \leq \lambda_i\}$. The *conjugate partition* of λ, denoted by λ', is a partition such that $Y_{\lambda'} = \{(i,j) : (j,i) \in Y_\lambda\}$. Finally, we write $\mu \subseteq \lambda$ if $Y_\mu \subseteq Y_\lambda$.

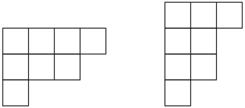

Figure: The Young diagrams of the partition $(4,3,1)$ and its conjugate partition $(3,2,2,1)$ often denoted as $(3,2^2,1)$.

Characters of symmetric groups

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ with *size* $\sum_{i=1}^{t} \lambda_i$. The *Young diagram* of λ is $Y_\lambda = \left\{(i,j) \in \mathbb{N}^2 : i \leq \ell(\lambda), j \leq \lambda_i\right\}$. The *conjugate partition* of λ, denoted by λ', is a partition such that $Y_{\lambda'} = \{(i,j) : (j,i) \in Y_\lambda\}$. Finally, we write $\mu \subseteq \lambda$ if $Y_\mu \subseteq Y_\lambda$.

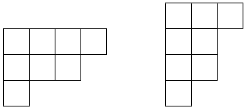

Figure: The Young diagrams of the partition $(4,3,1)$ and its conjugate partition $(3,2,2,1)$ often denoted as $(3,2^2,1)$.

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n.

Characters of symmetric groups

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ with *size* $\sum_{i=1}^{t} \lambda_i$. The *Young diagram* of λ is $Y_\lambda = \left\{(i,j) \in \mathbb{N}^2 : i \leq \ell(\lambda), j \leq \lambda_i\right\}$. The *conjugate partition* of λ, denoted by λ', is a partition such that $Y_{\lambda'} = \{(i,j) : (j,i) \in Y_\lambda\}$. Finally, we write $\mu \subseteq \lambda$ if $Y_\mu \subseteq Y_\lambda$.

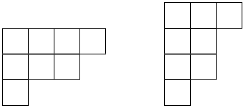

Figure: The Young diagrams of the partition $(4,3,1)$ and its conjugate partition $(3,2,2,1)$ often denoted as $(3,2^2,1)$.

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n. The trivial character of S_n is $\chi^{(n)}$, while the sign is $\chi^{(1^n)}$.

Characters of symmetric groups

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ with *size* $\sum_{i=1}^{t} \lambda_i$. The *Young diagram* of λ is $Y_\lambda = \left\{(i,j) \in \mathbb{N}^2 : i \leq \ell(\lambda), j \leq \lambda_i\right\}$. The *conjugate partition* of λ, denoted by λ', is a partition such that $Y_{\lambda'} = \{(i,j) : (j,i) \in Y_\lambda\}$. Finally, we write $\mu \subseteq \lambda$ if $Y_\mu \subseteq Y_\lambda$.

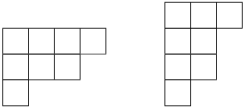

Figure: The Young diagrams of the partition $(4,3,1)$ and its conjugate partition $(3,2,2,1)$ often denoted as $(3,2^2,1)$.

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n. The trivial character of S_n is $\chi^{(n)}$, while the sign is $\chi^{(1^n)}$. The natural permutation character decomposes as $\chi^{(n)} + \chi^{(n-1,1)}$.

Multiplicity-free characters

We say that a character of group G is multiplicity-free if all its irreducible constituents appear with multiplicity one.

Multiplicity-free characters

We say that a character of group G is multiplicity-free if all its irreducible constituents appear with multiplicity one. Wildon in 2009 and independently Godsil and Meagher in 2010 classified all multiplicity-free permutation characters of S_n (with $n \geq 66$ for the former).

Multiplicity-free characters

We say that a character of group G is multiplicity-free if all its irreducible constituents appear with multiplicity one. Wildon in 2009 and independently Godsil and Meagher in 2010 classified all multiplicity-free permutation characters of S_n (with $n \geq 66$ for the former).

Anderson, Humphries and Nicholson have classified subgroups G of S_n such that for all irreducible characters ρ of G the character $\rho\big\uparrow_G^{S_n}$ is multiplicity-free.

Multiplicity-free characters

We say that a character of group G is multiplicity-free if all its irreducible constituents appear with multiplicity one. Wildon in 2009 and independently Godsil and Meagher in 2010 classified all multiplicity-free permutation characters of S_n (with $n \geq 66$ for the former).
Anderson, Humphries and Nicholson have classified subgroups G of S_n such that for all irreducible characters ρ of G the character $\rho\big\uparrow_G^{S_n}$ is multiplicity-free.

My goal is to look for pairs (G, ρ) of a subgroup G of S_n and its irreducible character ρ such that $\rho\big\uparrow_G^{S_n}$ is multiplicity-free.

Multiplicity-free characters

We say that a character of group G is multiplicity-free if all its irreducible constituents appear with multiplicity one. Wildon in 2009 and independently Godsil and Meagher in 2010 classified all multiplicity-free permutation characters of S_n (with $n \geq 66$ for the former). Anderson, Humphries and Nicholson have classified subgroups G of S_n such that for all irreducible characters ρ of G the character $\rho\big\uparrow_G^{S_n}$ is multiplicity-free.

My goal is to look for pairs (G, ρ) of a subgroup G of S_n and its irreducible character ρ such that $\rho\big\uparrow_G^{S_n}$ is multiplicity-free. We say that $G \leq S_n$ is *multiplicity-free* is there is such a pair (G, ρ).

Multiplicity-free characters

We say that a character of group G is multiplicity-free if all its irreducible constituents appear with multiplicity one. Wildon in 2009 and independently Godsil and Meagher in 2010 classified all multiplicity-free permutation characters of S_n (with $n \geq 66$ for the former).
Anderson, Humphries and Nicholson have classified subgroups G of S_n such that for all irreducible characters ρ of G the character $\rho\uparrow_G^{S_n}$ is multiplicity-free.

My goal is to look for pairs (G, ρ) of a subgroup G of S_n and its irreducible character ρ such that $\rho\uparrow_G^{S_n}$ is multiplicity-free. We say that $G \leq S_n$ is *multiplicity-free* is there is such a pair (G, ρ).
For example the group $S_k \times S_m \wr S_2 \leq S_{2m+k}$ is multiplicity-free.

Multiplicity-free characters

We say that a character of group G is multiplicity-free if all its irreducible constituents appear with multiplicity one. Wildon in 2009 and independently Godsil and Meagher in 2010 classified all multiplicity-free permutation characters of S_n (with $n \geq 66$ for the former).
Anderson, Humphries and Nicholson have classified subgroups G of S_n such that for all irreducible characters ρ of G the character $\rho\uparrow_G^{S_n}$ is multiplicity-free.

My goal is to look for pairs (G, ρ) of a subgroup G of S_n and its irreducible character ρ such that $\rho\uparrow_G^{S_n}$ is multiplicity-free. We say that $G \leq S_n$ is *multiplicity-free* is there is such a pair (G, ρ).
For example the group $S_k \times S_m \wr S_2 \leq S_{2m+k}$ is multiplicity-free. However, the permutation character of S_{2m+k} of the action on the cosets is not multiplicity-free.

Established classifications

All the multiplicity-free subgroups of S_n are now known.

Established classifications

All the multiplicity-free subgroups of S_n are now known.

Theorem (T, 2022), details omitted

Let $n \geq 66$. A subgroup $G \leq S_n$ is multiplicity-free if and only if it belongs to a list with certain groups of index $1, 2$ or 4 in $S_n, S_k \times S_l, S_k \times S_m \wr S_2, S_k \times S_2 \wr S_m, S_m \wr S_2, S_2 \wr S_m, S_m \wr S_3$ and $S_k \times L$ with L being one of $\mathrm{P\Gamma L}(2,8), \mathrm{ASL}(3,2), \mathrm{PGL}(2,5)$ and $\mathrm{AGL}(1,5)$.

Established classifications

All the multiplicity-free subgroups of S_n are now known.

Theorem (T, 2022), details omitted

Let $n \geq 66$. A subgroup $G \leq S_n$ is multiplicity-free if and only if it belongs to a list with certain groups of index $1, 2$ or 4 in $S_n, S_k \times S_l, S_k \times S_m \wr S_2, S_k \times S_2 \wr S_m, S_m \wr S_2, S_2 \wr S_m, S_m \wr S_3$ and $S_k \times L$ with L being one of $\mathrm{P\Gamma L}(2,8), \mathrm{ASL}(3,2), \mathrm{PGL}(2,5)$ and $\mathrm{AGL}(1,5)$.

There are some interesting restrictions for k, l and m in the full result.

Established classifications

All the multiplicity-free subgroups of S_n are now known.

Theorem (T, 2022), details omitted

Let $n \geq 66$. A subgroup $G \leq S_n$ is multiplicity-free if and only if it belongs to a list with certain groups of index $1, 2$ or 4 in $S_n, S_k \times S_l, S_k \times S_m \wr S_2, S_k \times S_2 \wr S_m, S_m \wr S_2, S_2 \wr S_m, S_m \wr S_3$ and $S_k \times L$ with L being one of $\mathrm{P\Gamma L}(2,8), \mathrm{ASL}(3,2), \mathrm{PGL}(2,5)$ and $\mathrm{AGL}(1,5)$.

There are some interesting restrictions for k, l and m in the full result. For instance the group $(S_k \times S_2 \wr S_m) \cap A_{k+2m}$ is multiplicity-free if and only if $k \geq 2m+2$.

Established classifications

All the multiplicity-free subgroups of S_n are now known.

Theorem (T, 2022), details omitted

Let $n \geq 66$. A subgroup $G \leq S_n$ is multiplicity-free if and only if it belongs to a list with certain groups of index $1, 2$ or 4 in $S_n, S_k \times S_l, S_k \times S_m \wr S_2, S_k \times S_2 \wr S_m, S_m \wr S_2, S_2 \wr S_m, S_m \wr S_3$ and $S_k \times L$ with L being one of $\mathrm{P\Gamma L}(2, 8), \mathrm{ASL}(3, 2), \mathrm{PGL}(2, 5)$ and $\mathrm{AGL}(1, 5)$.

There are some interesting restrictions for k, l and m in the full result. For instance the group $(S_k \times S_2 \wr S_m) \cap A_{k+2m}$ is multiplicity-free if and only if $k \geq 2m + 2$. Or the group $A_m \wr S_2$ embedded in S_{2m+1} is multiplicity-free if and only if m is a square.

Established classifications

All the multiplicity-free subgroups of S_n are now known.

Theorem (T, 2022), details omitted

Let $n \geq 66$. A subgroup $G \leq S_n$ is multiplicity-free if and only if it belongs to a list with certain groups of index $1, 2$ or 4 in $S_n, S_k \times S_l, S_k \times S_m \wr S_2, S_k \times S_2 \wr S_m, S_m \wr S_2, S_2 \wr S_m, S_m \wr S_3$ and $S_k \times L$ with L being one of $\mathrm{P\Gamma L}(2,8), \mathrm{ASL}(3,2), \mathrm{PGL}(2,5)$ and $\mathrm{AGL}(1,5)$.

There are some interesting restrictions for k, l and m in the full result. For instance the group $(S_k \times S_2 \wr S_m) \cap A_{k+2m}$ is multiplicity-free if and only if $k \geq 2m+2$. Or the group $A_m \wr S_2$ embedded in S_{2m+1} is multiplicity-free if and only if m is a square.

Along this result, for a majority of these groups G all the possible irreducible characters ρ with $\rho\uparrow_G^{S_n}$ multiplicity-free have been found.

Rectangular partitions

We say that λ is an (a,b)-birectangular partition if χ^λ is an irreducible constituent of $\left(\chi^{(a^b)} \boxtimes \chi^{(a^b)}\right) \uparrow_{S_{ab}\times S_{ab}}^{S_{2ab}}$.

Rectangular partitions

We say that λ is an (a, b)-birectangular partition if χ^λ is an irreducible constituent of $\left(\chi^{(a^b)} \boxtimes \chi^{(a^b)}\right) \uparrow_{S_{ab} \times S_{ab}}^{S_{2ab}}$. Note that λ is (a, b)-birectangular if and only if λ' is (b, a)-birectangular.

Proposition (T, 2022)

Let t be a non-negative integer. Suppose that $a > b$ are positive integers. Write $d = a - b$ and $0 \leq r \leq d - 1$ for the remainder of a modulo d. There is a partition λ of $2ab + t$ such that for some (a, b)-birectangular partitions μ and ν we have $\mu, \nu' \subseteq \lambda$ if and only if $t \geq 2r(d - r)$.

Rectangular partitions

We say that λ is an (a,b)-birectangular partition if χ^{λ} is an irreducible constituent of $\left(\chi^{(a^b)} \boxtimes \chi^{(a^b)}\right) \uparrow_{S_{ab} \times S_{ab}}^{S_{2ab}}$. Note that λ is (a,b)-birectangular if and only if λ' is (b,a)-birectangular.

Proposition (T, 2022)

Let t be a non-negative integer. Suppose that $a > b$ are positive integers. Write $d = a - b$ and $0 \leq r \leq d-1$ for the remainder of a modulo d. There is a partition λ of $2ab + t$ such that for some (a,b)-birectangular partitions μ and ν we have $\mu, \nu' \subseteq \lambda$ if and only if $t \geq 2r(d-r)$.

In particular, for $a > b$, there is a partition which is both (a,b)-birectangular and (b,a)-birectangular if and only if $b - a \mid a$.

Rectangular partitions

We say that λ is an (a, b)-birectangular partition if χ^{λ} is an irreducible constituent of $\left(\chi^{(a^b)} \boxtimes \chi^{(a^b)}\right) \uparrow_{S_{ab} \times S_{ab}}^{S_{2ab}}$. Note that λ is (a, b)-birectangular if and only if λ' is (b, a)-birectangular.

Proposition (T, 2022)

Let t be a non-negative integer. Suppose that $a > b$ are positive integers. Write $d = a - b$ and $0 \leq r \leq d - 1$ for the remainder of a modulo d. There is a partition λ of $2ab + t$ such that for some (a, b)-birectangular partitions μ and ν we have $\mu, \nu' \subseteq \lambda$ if and only if $t \geq 2r(d - r)$.

In particular, for $a > b$, there is a partition which is both (a, b)-birectangular and (b, a)-birectangular if and only if $b - a \mid a$. In fact one can show that there is a unique such partition given by $(2b^{2d}, 2(b-d)^{2d}, \ldots, (2d)^{2d})$ where $d = a - b$.

Rectangular partitions

We say that λ is an (a,b)-birectangular partition if χ^λ is an irreducible constituent of $\left(\chi^{(a^b)} \boxtimes \chi^{(a^b)}\right) \uparrow_{S_{ab}\times S_{ab}}^{S_{2ab}}$. Note that λ is (a,b)-birectangular if and only if λ' is (b,a)-birectangular.

Proposition (T, 2022)

Let t be a non-negative integer. Suppose that $a > b$ are positive integers. Write $d = a - b$ and $0 \leq r \leq d-1$ for the remainder of a modulo d. There is a partition λ of $2ab + t$ such that for some (a,b)-birectangular partitions μ and ν we have $\mu, \nu' \subseteq \lambda$ if and only if $t \geq 2r(d-r)$.

In particular, for $a > b$, there is a partition which is both (a,b)-birectangular and (b,a)-birectangular if and only if $b - a \mid a$. In fact one can show that there is a unique such partition given by $(2b^{2d}, 2(b-d)^{2d}, \ldots, (2d)^{2d})$ where $d = a - b$.

Rectangular partitions

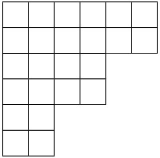

Figure: The unique partition $(6^2, 4^2, 2^2)$ which is both $(4, 3)$-birectangular and $(3, 4)$-birectanuglar.

Rectangular partitions

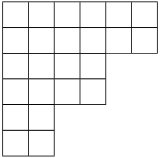

Figure: The unique partition $(6^2, 4^2, 2^2)$ which is both $(4,3)$-birectangular and $(3,4)$-birectanuglar.

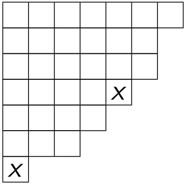

Figure: A partition $(7, 6^2, 5, 4, 3, 1)$ of size 32 which contains both a $(5,3)$-birectangular and a $(3,5)$-birectanuglar partition.

Rectangular partitions

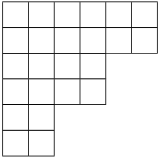

Figure: The unique partition $(6^2, 4^2, 2^2)$ which is both $(4, 3)$-birectangular and $(3, 4)$-birectanuglar.

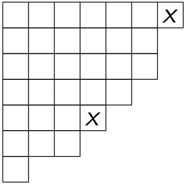

Figure: A partition $(7, 6^2, 5, 4, 3, 1)$ of size 32 which contains both a $(5, 3)$-birectangular and a $(3, 5)$-birectanuglar partition.