TOWARDS ODD-SUNFLOWERS: TEMPERATE FAMILIES AND LIGHTNINGS

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1. Introduction

For integer $n \geq 0$, let $[n] = \{1, \ldots, n\}$. We use $\mathcal{P}(S)$ to denote all subsets of S and $\binom{S}{k}$ to denote all subsets of S of size k. Finally, for $\mathcal{F} \subseteq \mathcal{P}(S)$, we write $\mathcal{F}_k$ for $\mathcal{F} \cap \binom{S}{k}$.

A set family $\mathcal{F} \subseteq \mathcal{P}([n])$ is called <u>t-temperate</u> if for all $A \in \mathcal{F}$, $|\mathcal{P}(A) \cap \mathcal{F}| \leq \sum_{i=0}^{t} \binom{|A|}{i}$.

Example 1.1. Some examples of t-temperate families are:

(1) A set family is 0-temperate if and only if it is an antichain.
(2) A maximal chain is t-temperate for $t \geq 1$.
(3) For any integer $k \geq 0$, $\bigcup_{i=k}^{k+t} \binom{[n]}{i}$ is t-temperate.

Question 1: What is the maximal size of a t-temperate set family on $[n]$? — *fully answered*

Question 2: What is the maximal size of an intersecting t-temperate set family on $[n]$? — *answered for $t = 1$ and odd n; conjectured for $t = 1$ and even n*

2. Motivation

Our motivation comes from odd-sunflowers, which, in turn, arise as a variant of sunflowers.

An <u>r-sunflower</u> is a set family $S_1, \ldots, S_r$ such that for $i \neq j$, $S_i \cap S_j = \bigcap_{k=1}^{r} S_k$.

Conjecture 2.1 (Erdős, Rado '60)**.** *There is $c = c(r)$ such that any set family $\mathcal{F}$ of sets of size at most w with no r-sunflower satisfies $|\mathcal{F}| < c^w$.*

A recent breakthough by Alweiss, Lovett, Wu and Zhang in 2021 leads to the best known upper bound on $|\mathcal{F}|$ at the moment due to Fukuyama — $O(r^2 \log w / \log\log w)^w$.

A different direction for sunflowers aims to understand $f(n)$ defined as the maximal size of $\mathcal{F} \subseteq \mathcal{P}([n])$ without sunflowers ($=$ r-sunflowers for all $r \geq 3$).

Date: November 6, 2025.

Conjecture 2.2 (Erdős, Szemerédi '78)**.** $\phi := \lim_{n\to\infty} f(n)^{1/n} < 2$.

In 2016, Alon, Shpilka and Umans proved that this conjecture is equivalent to showing that subsets of $\mathbb{F}_3^n$ containing no cap set have size $O(3^{(1-\varepsilon)n})$ for some $\varepsilon > 0$. This was ultimately proved in 2017 by Ellenberg and Gijswijt, building on the techniques of Croot, Lev and Pach, establishing the conjecture.

Now: $1.551 < \phi < 1.890$ due to Deuber, Erdős, Gunderson, Kostochka and Meyer (lower bound), and Naslund and Sawin (upper bound).

The following variants of sunflowers were introduced by Frankl, Pach and Pálvölgyi in 2024. A non-empty set family is an even-sunflower, if every element is contained in an even number of sets (or in none). Analogously, a family of at least two sets is an odd-sunflower, if every element is contained in an odd number of sets, or in none.

Note that $(2s + 1)$-sunflowers are odd-sunflowers, but that is the only relation between odd-, even- and r-sunflowers.

Let $f_{\text{even}}(n)$ and $f_{\text{odd}}(n)$ be the maximum size of a set family on $[n]$ with no even-sunflower, resp., odd-sunflower. These two quantities exhibit a diametrically opposite behaviour. A version of 'odd-town' problem gives.

Theorem 2.3 (Frankl, Pach, Pálvölgyi '24)**.** $f_{\text{even}}(n) = n$.

On the other hand.

Theorem 2.4 (Frankl, Pach, Pálvölgyi '24)**.** $f_{\text{odd}}(n) > 1.502^n$.

The best upper-bound comes from $f_{\text{odd}}(n) < \phi^n < 1.890^n$. The motivation to look at intersecting 1-temperate families comes from the following.

Lemma 2.5. *A set family with no odd-sunflowers is an intersecting* 1-*temperate family.*

Proof. Intersecting: any two disjoint sets form an odd-sunflower.

1-temperate: For A in our set family, its subsets in the family do not form an odd-sunflower and so its proper subsets in the family do not form an even-sunflower. We apply Theorem 2.3 with $n = |A|$. □

3. t-TEMPERATE FAMILIES

Theorem 3.1. *Let* $0 \le t \le n$ *be integers. The maximum size of a* t-*temperate family on* $[n]$ *is* $\sum_{j=0}^{t} \binom{n}{\lfloor (n-t)/2 \rfloor + j}$.

This specialises to Sperner's theorem when $t = 0$.

Theorem 3.2 (Sperner's theorem)**.** *Let* $n \ge 0$ *be an integer. The maximum size of an antichain on* $[n]$ *is* $\binom{n}{\lfloor n/2 \rfloor}$.

(Sketch proof). The antichain $\binom{[n]}{\lfloor n/2 \rfloor}$ attains the cardinality. Now, let $\mathcal{C}$ be a maximal chain on $[n]$ chosen uniformly at random. Then

$$1 \geq \mathbb{E}_{\mathcal{C}}[|\mathcal{C} \cap \mathcal{F}|] = \sum_{i=0}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{i}}.$$

As $\binom{n}{\lfloor n/2 \rfloor}$ is the largest binomial coefficient $\binom{n}{i}$ the result follows. □

Maybe $\mathbb{E}_{\mathcal{C}}[|\mathcal{C} \cap \mathcal{F}|] \leq t+1$ for t-temperate families — unfortunately not (consider $t = 1$ and a maximal chain). Instead we use a weighted argument: the weight of $A \in \mathcal{F}$ is $w(A) = \binom{n}{|A|}$, and the weight of a family $\mathcal{F}$ is $w(\mathcal{F}) = \sum_{A \in \mathcal{F}} w(A)$.

Lemma 3.3. *Let $\mathcal{F}$ be (any) family on $[n]$ and let $\mathcal{C}$ be a random maximal chain on $[n]$ chosen uniformly at random. Then $\mathbb{E}_{\mathcal{C}}[w(\mathcal{F} \cap \mathcal{C})] = |\mathcal{F}|$.*

Proof. We simply compute $\mathbb{E}_{\mathcal{C}}[w(\mathcal{F} \cap \mathcal{C})] = \sum_{i=0}^{n} w([i]) \cdot \mathbb{P}_{\mathcal{C}}(\mathcal{F}_i \cap \mathcal{C} \neq \emptyset) = \sum_{i=0}^{n} w([i]) \cdot |\mathcal{F}_i| / \binom{n}{i} = \sum_{i=0}^{n} |\mathcal{F}_i| = |\mathcal{F}|$. □

To use the t-temperate condition, we want to look at the maximal elements in $\mathcal{C}$. Let $\mathcal{F}^*$ be the set of $A \in \mathcal{F}$ which are maximal in some maximal chain on $[n]$. Clearly, all maximal elements of $\mathcal{F}$ lie in $\mathcal{F}^*$.

Proof of Theorem 3.1. The t-temperate family $\bigcup_{j=0}^{t} \binom{[n]}{\lfloor (n-t)/2 \rfloor + j}$ has size equal to $\sum_{j=0}^{t} \binom{n}{\lfloor (n-t)/2 \rfloor + j}$. Let $\mathcal{F}$ be a (non-empty) t-temperate family on $[n]$ and let $\mathcal{C}$ be a maximal chain on $[n]$ chosen uniformly at random. Define $M(\mathcal{C}) = \max \mathcal{C} \cap \mathcal{F}$ when the intersection is non-empty, and $M(\mathcal{C}) = \star$ otherwise.

Applying Lemma 3.3 and using conditional expectation (with the convention that $\mathbb{E}[X \mid Y] = \mathbb{P}(U \mid Y) = 0$ when $\mathbb{P}(Y) = 0$), we have

$$\begin{aligned}
|\mathcal{F}| &= \mathbb{E}_{\mathcal{C}}\Big[w(\mathcal{C} \cap \mathcal{F})\Big] \\
&= \mathbb{E}_{\mathcal{C}}\Big[w(\mathcal{C} \cap \mathcal{F}) \,\Big|\, M(\mathcal{C}) = \star\Big] \cdot \mathbb{P}_{\mathcal{C}}\Big(M(\mathcal{C}) = \star\Big) + \sum_{A \in \mathcal{F}^*} \mathbb{E}_{\mathcal{C}}\Big[w(\mathcal{C} \cap \mathcal{F}) \,\Big|\, M(\mathcal{C}) = A\Big] \cdot \mathbb{P}_{\mathcal{C}}\Big(M(\mathcal{C}) = A\Big) \\
&= \sum_{A \in \mathcal{F}^*} \mathbb{E}_{\mathcal{C}}\Big[w(\mathcal{C} \cap \mathcal{F}) \,\Big|\, M(\mathcal{C}) = A\Big] \cdot \mathbb{P}_{\mathcal{C}}\Big(M(\mathcal{C}) = A\Big) \\
&\leq \max_{A \in \mathcal{F}^*} \mathbb{E}_{\mathcal{C}}\Big[w(\mathcal{C} \cap \mathcal{F}) \,\Big|\, M(\mathcal{C}) = A\Big] \\
&= \max_{A \in \mathcal{F}^*} \sum_{B \in \mathcal{F} \cap \mathcal{P}(A)} w(B) \cdot \mathbb{P}_{\mathcal{C}}\Big(B \in \mathcal{C} \,\Big|\, M(\mathcal{C}) = A\Big) \\
&= \max_{A \in \mathcal{F}^*} \sum_{B \in \mathcal{F} \cap \mathcal{P}(A)} \binom{n}{|B|} \Big/ \binom{|A|}{|B|}.
\end{aligned}$$

Note that $\binom{n}{|B|} / \binom{|A|}{|B|} = \binom{n}{|A|} / \binom{n-|B|}{n-|A|}$, and thus is non-decreasing as $|B|$ grows from 0 to $|A|$. Using the condition $|\mathcal{P}(A) \cap \mathcal{F}| \leq \sum_{i=0}^{t} \binom{|A|}{i}$, we find

$$
\begin{aligned}
|\mathcal{F}| &\le \max_{A\in\mathcal{F}^*} \sum_{B\in\mathcal{F}\cap\mathcal{P}(A)} \binom{n}{|B|} \Big/ \binom{|A|}{|B|} \\
&\le \max_{A\in\mathcal{F}^*} \sum_{j=\max(0,|A|-t)}^{|A|} \binom{n}{j} \\
&\le \sum_{j=0}^{t} \binom{n}{\lfloor (n-t)/2\rfloor + j},
\end{aligned}
$$

□

Equality cases:

(a) For $t \equiv n \pmod 2$, we get $\bigcup_{j=0}^{t} \binom{[n]}{(n-t)/2+j}$.
(b) For $t = n-1$, we can take any family on $[n]$ of size $2^n - 1$.
(c) If $t \not\equiv n \pmod 2$, we get $\bigcup_{j=0}^{t} \binom{[n]}{(n-t-1)/2+j}$ and $\bigcup_{j=0}^{t} \binom{[n]}{(n-t+1)/2+j}$.

4. Intersecting 1-temperate families

Theorem 4.1. *The maximum size of an intersecting 1-temperate family on $[n]$ for odd $n = 2k-1$ is $\binom{2k-1}{k} + \binom{2k-1}{k+1}$.*

Recall the wrong assertion $\mathbb{E}_{\mathcal{C}}[|\mathcal{C}\cap\mathcal{F}|] \le t+1$ for t-temperate $\mathcal{F}$. This can be easily fixed for $t = 1$.

Lemma 4.2. *Let $\mathcal{C}$ be a maximal chain on $[n]$ chosen uniformly at random. If $\mathcal{F}$ is a 1-temperate family on $[n]$ which does not contain the empty set, then $\mathbb{E}_{\mathcal{C}}[|\mathcal{C}\cap\mathcal{F}|] \le 2$.*

Proof. Assume $\mathcal{F}$ is non-empty and let $E = \mathbb{E}_{\mathcal{C}}[|\mathcal{C}\cap\mathcal{F}|]$. Reusing the notation $M(\mathcal{C})$, we have

$$
\begin{aligned}
E &= \mathbb{E}_{\mathcal{C}}\Big[|\mathcal{C}\cap\mathcal{F}| \,\Big|\, M(\mathcal{C}) = \star\Big] \cdot \mathbb{P}_{\mathcal{C}}\Big(M(\mathcal{C}) = \star\Big) + \sum_{A\in\mathcal{F}^*} \mathbb{E}_{\mathcal{C}}\Big[|\mathcal{C}\cap\mathcal{F}| \,\Big|\, M(\mathcal{C}) = A\Big] \cdot \mathbb{P}_{\mathcal{C}}\Big(M(\mathcal{C}) = A\Big) \\
&= \sum_{A\in\mathcal{F}^*} \mathbb{E}_{\mathcal{C}}\Big[|\mathcal{C}\cap\mathcal{F}| \,\Big|\, M(\mathcal{C}) = A\Big] \cdot \mathbb{P}_{\mathcal{C}}\Big(M(\mathcal{C}) = A\Big) \\
&\le \max_{A\in\mathcal{F}^*} \mathbb{E}_{\mathcal{C}}\Big[|\mathcal{C}\cap\mathcal{F}| \,\Big|\, M(\mathcal{C}) = A\Big] \qquad (1) \\
&= 1 + \max_{A\in\mathcal{F}^*} \sum_{\substack{B\in\mathcal{F}\cap\mathcal{P}(A)\\ B\ne A}} \mathbb{P}_{\mathcal{C}}\Big(B\in\mathcal{C} \,\Big|\, M(\mathcal{C}) = A\Big) \\
&= 1 + \max_{A\in\mathcal{F}^*} \sum_{\substack{B\in\mathcal{F}\cap\mathcal{P}(A)\\ B\ne A}} \frac{1}{\binom{|A|}{|B|}}.
\end{aligned}
$$

Since $\mathcal{F}$ does not contain the empty set and $|\mathcal{F}\cap\mathcal{P}(A)| \le |A|+1$ for any A in $\mathcal{F}$, we have

$$
\begin{aligned}
E &\le 1 + \max_{A\in\mathcal{F}^*} \sum_{\substack{B\in\mathcal{F}\cap\mathcal{P}(A)\\ B\neq A}} \frac{1}{\binom{|A|}{|B|}} \\
&\le 1 + \max_{A\in\mathcal{F}^*} \sum_{\substack{B\in\mathcal{F}\cap\mathcal{P}(A)\\ B\neq A}} \frac{1}{|A|} \qquad (2)\\
&\le 2, \qquad (3)
\end{aligned}
$$

as required. □

One can now proceed as for Sperner's theorem.

Proof of Theorem 4.1. The cardinality is attained by $\binom{[2k-1]}{k}\cup\binom{[2k-1]}{k+1}$ (which is intersecting 1-temperate). Now let $\mathcal{F}$ be an intersecting 1-temperate family (so $\emptyset \notin \mathcal{F}$). By Lemma 4.2 we have

$$2 \ge \mathbb{E}_{\mathcal{C}}[|\mathcal{C}\cap\mathcal{F}|] = \sum_{i=0}^{2k-1} \frac{|\mathcal{F}_i|}{\binom{n}{i}}.$$

As $\binom{2k-1}{k} = \binom{2k-1}{k-1}$ and $\binom{2k-1}{k+1} = \binom{2k-1}{k-2}$ are the largest 4 binomial coefficients $\binom{n}{i}$, and we can only have half of elements from $\binom{[2k-1]}{k} \cup \binom{[2k-1]}{k-1}$, the result follows. □

Equality cases:

(a) For $n \neq 3$, we get $\binom{[2k-1]}{k} \cup \binom{[2k-1]}{k+1}$.

(b) For $n = 3$, we get $\binom{[3]}{2} \cup \binom{[3]}{3}$ and $\{A \in \mathcal{P}([3]) : i \in A\}$ for each $i \in [3]$.

For even n, we make the following conjecture.

Conjecture 4.3. *The maximum size of an intersecting* 1*-temperate family on* $[n]$ *for even* $n = 2k$ *is* $\frac{1}{2}\binom{2k}{k} + \binom{2k}{k+1} + \binom{2k-1}{k+2}$.

This is among others attained by the lightning: the family on $[2k]$ which contains:

- all elements of $\binom{[2k]}{k}$ containing 1, and
- all elements of $\binom{[2k]}{k+1}$, and
- all elements of $\binom{[2k]}{k+2}$ not containing 1.