Plethystic Murnaghan–Nakayama rule: proof using Loehr's labelled abacus

Pavel Turek

Royal Holloway, University of London

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Based on arXiv:2312.02958

Outline

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1. Symmetric functions
2. Labelled abacus
3. Plethystic Murnaghan–Nakayama rule

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We write $\mathbb{Q}\Lambda = \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$ for the ring of symmetric functions over rational numbers.

Connection to representation theory

$$
\begin{tikzcd}
& \Lambda \arrow[dl, leftrightarrow, "\text{Formal characters}"'] \arrow[dr, leftrightarrow, "\text{Frobenius map}"] & \\
\text{PolyGL-mod} \arrow[rr, leftrightarrow, "\text{Schur functors}"'] & & \text{Sym-mod}
\end{tikzcd}
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PolyGL-mod stands for polynomial representations of general linear group.

Sym-mod denotes the representations of symmetric groups.

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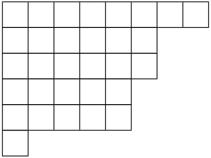

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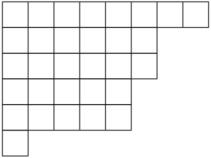

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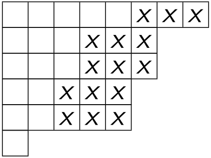

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Example: The crossed boxes form the Young diagram of $(8,6,6,5,5,1)/(5,3,3,2,2,1)$.

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- Other bases such as monomial symmetric functions, elementary symmetric functions, ...

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Schur functions correspond to ordinary irreducible representation.

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Plethysms correspond to induced characters from ‘maximal’ imprimitive subgroups, called wreath products of symmetric groups. They are difficult to understand, even the decomposition of $s_{(a)} \circ s_{(b)}$ as a sum of Schur functions is not known in general.

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Generalisations of labelled abaci have appeared since 2010. For example, abacus tournaments model Hall–Littlewood polynomials.

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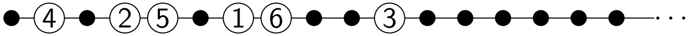

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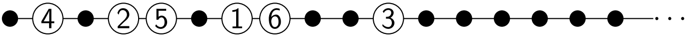

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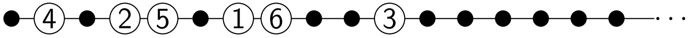

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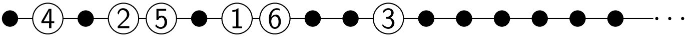

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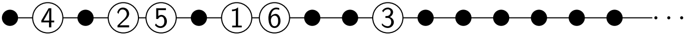

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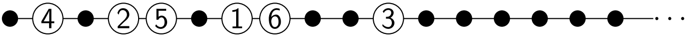

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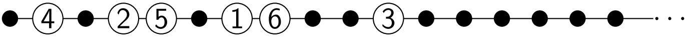

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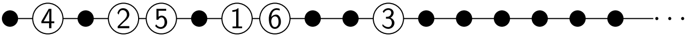

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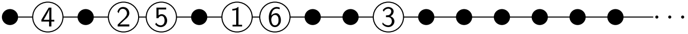

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Movement of beads

The key identity is $a_{\lambda+\delta(N)} = \sum_{w \in \mathrm{Abc}_N(\lambda)} \mathrm{sgn}(w)\,\mathrm{wt}(w)$, where λ has length at most N and $\mathrm{Abc}_N(\lambda)$ is the set of labelled abaci with N beads and shape λ.

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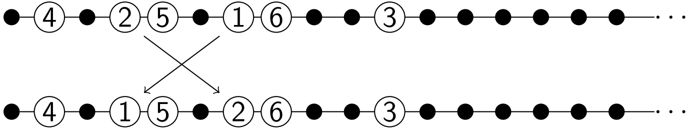

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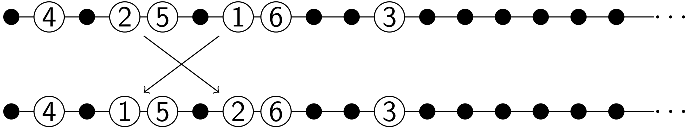
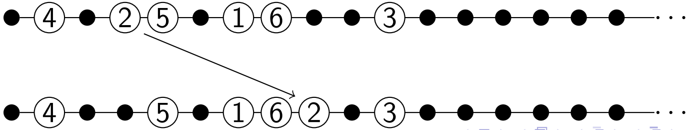


- Moving bead to an empty position: The shape changes. The sign changes if and only if the bead 'jumps over' odd number of beads.



Outline

Border strips

A connected skew partition with no 2×2 square is called border strip.

			X	
			X	
		X	X	
		X		

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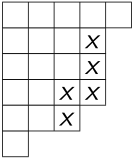

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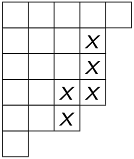

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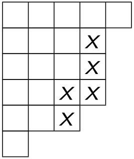

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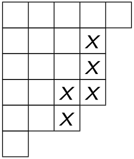

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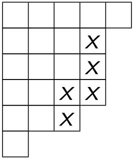
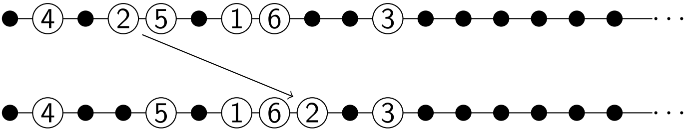
Border strips - correspondence

The border strip



corresponds to the move



r-decomposable partitions

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					c	c	c
			a	b	c		
			a	b	c		
		a	a	b			
		a	b	b			

Example: The skew partition $(8,6,6,5,5,1)/(5,3,3,2,2,1)$ is 5-decomposable (with $d = 3$). The three border strips are labelled by a, b and c, respectively.

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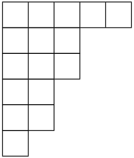

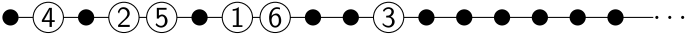

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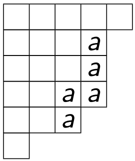
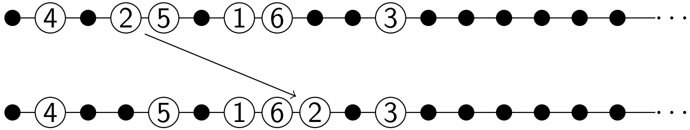
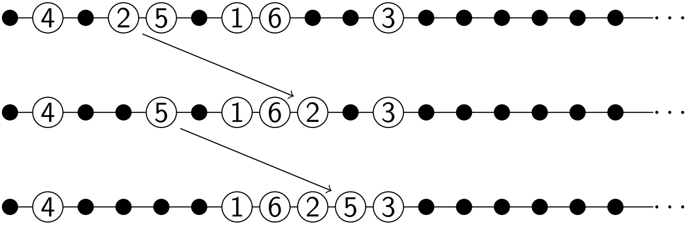




r-decomposable partitions - correspondence

			a	b
			a	b
		a	a	b
		a	b	b



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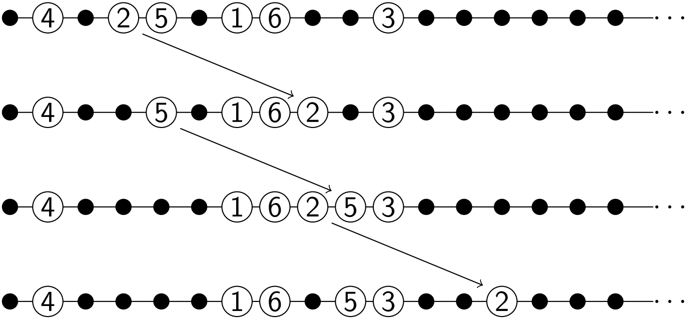
					c	c	c
			a	b	c		
			a	b	c		
		a	a	b			
		a	b	b			



Statement

Theorem (Plethystic Murnaghan–Nakayama rule)

Let μ be a partition and r and m be positive integers. Then

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By repeated applications of the rule, one obtains the decomposition of $s_\mu(p_\rho \circ h_\nu)$.

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Case $m = 1$ - example

$$\sum_{i=1}^{N} \sum_{w \in \mathrm{Abc}_N(\mu)} \mathrm{sgn}(w)\,\mathrm{wt}(w) x_i^r = \sum_{\mu \subseteq \lambda} \sum_{w \in \mathrm{Abc}_N(\lambda)} \mathrm{sgn}(\lambda/\mu)\,\mathrm{sgn}(w)\,\mathrm{wt}(w)$$

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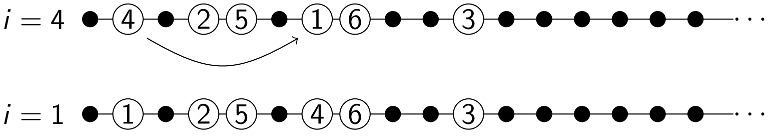
Cancellation:



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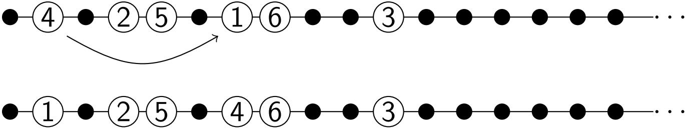
$$\sum_{i=1}^{N} \sum_{w \in \mathrm{Abc}_N(\mu)} \operatorname{sgn}(w)\,\mathrm{wt}(w)\,x_i^r = \sum_{\mu \subseteq \lambda} \sum_{w \in \mathrm{Abc}_N(\lambda)} \operatorname{sgn}(\lambda/\mu)\,\operatorname{sgn}(w)\,\mathrm{wt}(w)$$

In the example $r = 5$.

Cancellation:

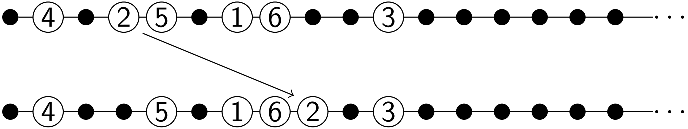
$i = 4$

$i = 1$



Successful move:

$i = 2$



General case

If $\mathrm{Com}_N(m)$ is the set of compositions of m of length N, this time we need to show

$$\sum_{\substack{w \in \mathrm{Abc}_N(\mu) \\ \beta \in \mathrm{Com}_N(m)}} \mathrm{sgn}(w)\,\mathrm{wt}(w) x_1^{r\beta_1} x_2^{r\beta_2} \cdots x_N^{r\beta_N}$$
$$= \sum_{\mu \subseteq \lambda} \sum_{w \in \mathrm{Abc}_N(\lambda)} \mathrm{sgn}(\lambda/\mu)\,\mathrm{sgn}(w)\,\mathrm{wt}(w),$$

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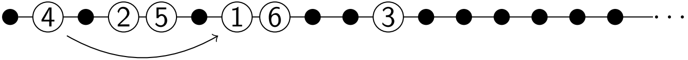
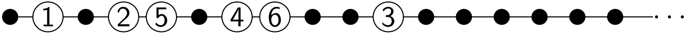
This time for a given pair (β, w) we read the abacus rightwards and move bead i (by r places rightwards) β_i-times. There are again two cases.

General case - cancellation

$$\sum_{\substack{w \in \mathrm{Abc}_N(\mu) \\ \beta \in \mathrm{Com}_N(m)}} \mathrm{sgn}(w)\,\mathrm{wt}(w) x_1^{r\beta_1} x_2^{r\beta_2} \cdots x_N^{r\beta_N}$$

$$= \sum_{\mu \subseteq \lambda} \sum_{w \in \mathrm{Abc}_N(\lambda)} \mathrm{sgn}(\lambda/\mu)\,\mathrm{sgn}(w)\,\mathrm{wt}(w)$$

$\beta = (0, 2, 0, 1, 0, 0)$ 

$\beta = (1, 2, 0, 0, 0, 0)$ 

General case - successful moves

$$\sum_{\substack{w\in \mathrm{Abc}_N(\mu)\\ \beta\in \mathrm{Com}_N(m)}} \mathrm{sgn}(w)\,\mathrm{wt}(w) x_1^{r\beta_1} x_2^{r\beta_2}\cdots x_N^{r\beta_N}$$

$$= \sum_{\mu\subseteq\lambda}\ \sum_{w\in \mathrm{Abc}_N(\lambda)} \mathrm{sgn}(\lambda/\mu)\,\mathrm{sgn}(w)\,\mathrm{wt}(w)$$

$\beta = (0, 2, 0, 0, 1, 0)$

$\beta = (0, 1, 0, 0, 1, 0)$

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