Decomposition columns labelled by d-balanced partitions

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October 7, 2025

joint work with David Hemmer, Bim Gustavsson and Stacey Law

Young diagrams

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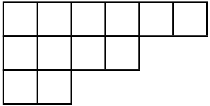

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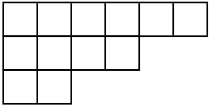

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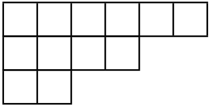

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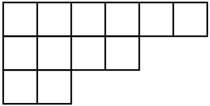

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Example: Partition $(7, 7, 4, 3, 3, 3, 3, 2)$ is 5-regular but not 3-regular.

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The irreducible composition factors of the Specht modules in (quantum) characteristic e are given by *the decomposition matrix*.

Decomposition matrix

$S^{(\cdot)}$ \ $D^{(\cdot)}$	6	$5,1$	$4,2$	$3,3$	$4,1,1$	$3,2,1$	$2,2,1,1$
6	1						
$5,1$	1	1					
$4,2$			1				
$3,3$		1		1			
$4,1,1$		1			1		
$3,2,1$	1	1		1	1	1	
$2,2,2$	1					1	
$3,1,1,1$					1	1	
$2,2,1,1$							1
$2,1,1,1,1$				1		1	
$1,1,1,1,1,1$				1			

Table: The decomposition matrix of S_6 in characteristic $e = 3$.

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$4,1,1$		1			1		
$3,2,1$	1	1		1	1	1	
$2,2,2$	1					1	
$3,1,1,1$					1	1	
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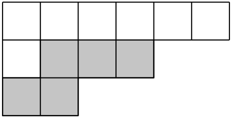

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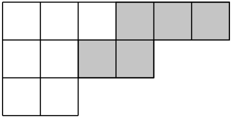

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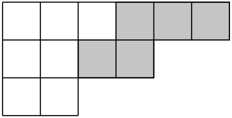

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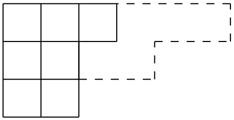

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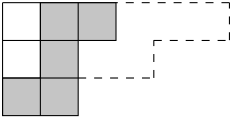

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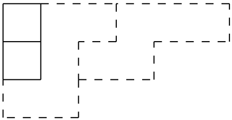

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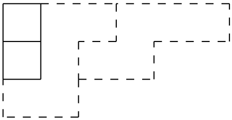

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Definition

Let d be a positive integer. We say that a partition λ is *d-balanced* if d divides all arm lengths of its e-divisible hooks.

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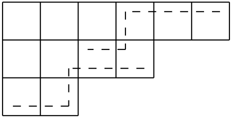

Figure: The partition $(6, 4, 2)$ is 3-balanced

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3	4				

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Let d be a positive integer. The *d-runner matrix* of λ stores in row x $(1 \leq x \leq d - 1)$ and column y $(0 \leq y \leq e - 1)$ the number of parts of λ which have remainder x modulo d and their rightmost box has e-residue y.

	$\cdots$	y

$$\longleftrightarrow$$
$$x \pmod{d}$$

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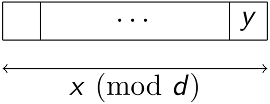

The partition $(6, 4, 2)$ drawn above has 3-runner matrix $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

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Let $1 < d < e$ be coprime integers and μ be an e-regular d-balanced partition of some integer n. We write γ for the e-core of μ and $\mathcal{R}$ for the d-runner matrix of μ.

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Theorem (Gustavsson, Law, T. 2025+)

Let d, e, μ, γ and $\mathcal{R}$ be as above. If μ satisfies Condition (1), then the non-zero decomposition numbers in the column labelled by μ are 1 and lie precisely in rows labelled by partitions with e-core γ and d-runner matrix $\mathcal{R}$.

Conjecture

Our theorem can be rephrased as: Condition (1) (for all μ) is equivalent to

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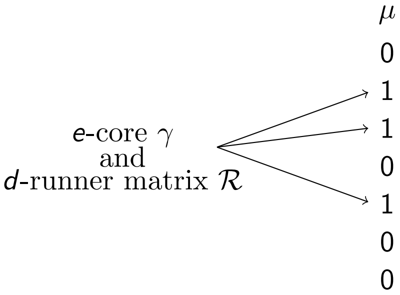

Figure: The decomposition column labelled by e-regular and d-balanced μ.

Steps towards the conjecture

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4. If γ is obtained from μ by removing at most 3 hooks of size e, the conjecture holds for μ. This follows from a result of Fayers.
5. (T. 2025+) The top and bottom non-zero entries of the decomposition column labelled by μ agree with the conjecture.

Peek into the combinatorics

Theorem (T. 2025+)

Let $1 < d < e$ be coprime integer. The map

$$\phi : \{d\text{-balanced partitions}\} \to \{e\text{-cores}\} \times \mathbb{Z}_{\geq 0}^{(d-1)\times e}$$

given by taking the e-core and the d-runner matrix of a partition is injective. Its left inverse is given by mapping $(\gamma, \mathcal{R})$ to the lex-maximal size-minimal partition with e-core γ and d-runner matrix $\mathcal{R}$.

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Furthermore, a d-balanced partition μ is e-regular if there is 0 in the first row of its d-runner matrix. Thus, at least for $d = 2$, there are many e-regular d-balanced partitions.