FUN (SOMEWHAT) ELEMENTARY ENCOUNTERED QUESTIONS

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Here is a collection of some of the small questions encountered during my research journey which stood out due to their elegance, beauty, simplicity (of their statement), or I just enjoyed solving them. At the moment, this document contains only the statements of the questions and reference to where one can find solutions (which are often written in more complex research language). Hopefully, one day, more comprehensive solutions will appear here.

As a guidance, each problem is assigned from one (= easy) to five (= difficult) stars, indicating its subjective, relative difficulty. If you find an easy intuitive solution to one of the more difficult questions, I have likely not thought of it and would thus be grateful to hear it!

Problems

1. ($\star\star$) Let $d, e > 1$ be two integers. On a line, there are comfortable patches of grass labelled by positive integers, finitely many of which are occupied by a hare. Each patch of grass can be occupied by only one hair at a time and each hare can leap from its current patch p to a vacant patch q if $e \mid q - p$. Each hare has between 0 and $d-1$ teeth and is *happy* if it sits on a patch of grass p such that the number of vacant patches of grasses r with $r \leq p$ is congruent to the hare's number of teeth modulo d.

   We say that an arrangement of hares on the patches is *balanced* if for any $p < q$ with $e \mid q - p$ such that patch p is vacant and patch q is occupied, the number of vacant patches r with $p \leq r \leq q$ is not divisible by d. If we do not distinguish between hares with the same number of teeth, show that there is at most one balanced arrangement which can be reached from the original arrangement through a finite number of leaps and makes all hares happy. An example of such an arrangement is in Figure 1.

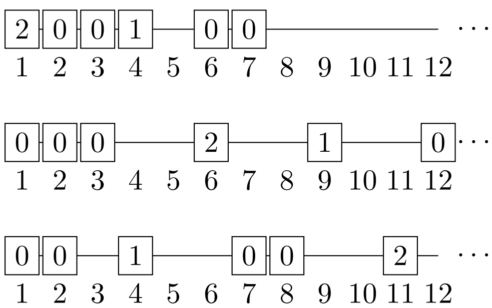

Figure 1. Let $d = 3$ and $e = 5$. The first line shows an initial position of hares — boxes filled with a number indicating the number of teeth of the given hare. The second line shows a balanced arrangement with each hare happy (obtained from the initial arrangement through a series of leaps; for instance: hare at 7 leaps to 12, hare at 4 leaps to 9, and hares at 1 and 6 switch their position, using, for example, patch 11). The final line then shows a strictly balanced arrangement with each hare happy.

2. ($\star\star\star$) Assuming that d and e are coprime in the previous question, show that the hares can always reach a balanced arrangement which makes all of them happy.

Date: August 31, 2026.

3. (⋆⋆⋆⋆⋆) Still assuming that d and e are coprime in the previous questions, we say that an arrangement of hares on the patches is *strictly balanced* if for any $p < q$ with $e \mid q - p$ such that patch p is vacant and patch q is occupied, the number of vacant patches r with $p < r < q$ is not divisible by d. If we do not distinguish between hares with the same number of teeth, show that there is at most one strictly balanced arrangement which can be reached from the original arrangement through a finite number of leaps and makes all hares happy. An example of such an arrangement is in Figure 1.

4. (⋆⋆⋆⋆) Let n and N be positive integers. During an N-day long *Sun and Flower camp*, n children have a chance to practice making lunches in different groups. Each day, exactly one lunch is made by a group of children that have not cooked together in the same formation before. After the camp is over the children notice an intriguing property of their cooking arrangements: if a given group of k children cooked a lunch one of the N days, then at most k *other* lunches were cooked by some of these k children and no other children. Show that $N \leq \binom{n}{\lfloor (n-1)/2 \rfloor} + \binom{n}{\lfloor (n+1)/2 \rfloor}$.

5. (⋆⋆⋆) Leo has two equal rectangular pieces of paper which are divided by drawn lines into $a \times b$ grids of unit squares with $a < b$ positive integers. Leo performs the following operation. The two papers are placed next to each other, one of the left and the other on the right so that they share an edge of length a. Leo then cuts along some north-east path from the south-west corner to the north-east corner of the right rectangle (the path consists of segments of the border and drawn lines). Leo then rotates the cut out part of the right rectangle containing its south-east corner (which may be the whole rectangle) by $180°$ through the south-west corner of this rectangle and glues shared edges together; see Figure 2 for an example.

   Once this is done, the shape Leo obtains is symmetric about the south-east line ℓ going through the north-west corner of the left rectangle (as in Figure 2). Show that $b - a \mid a$. (What does the shape Leo obtains look like?) If instead Leo performs such an operation twice and obtains two shapes which are reflections of each other over line ℓ, is it still true that $b - a$ must divide a?

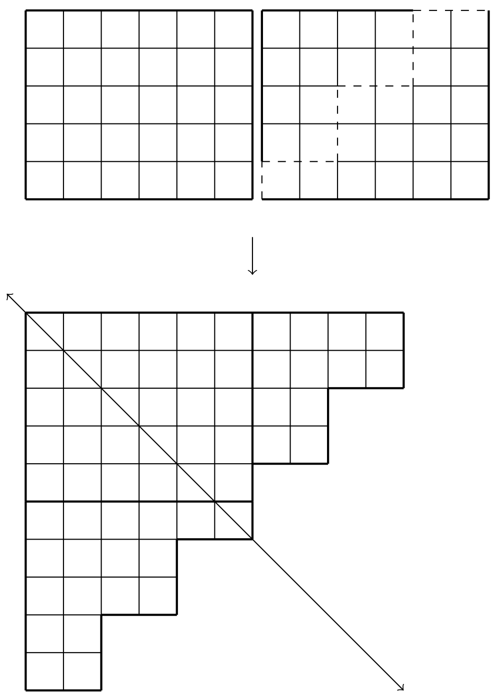

FIGURE 2. For $a = 5$ and $b = 6$ Leo can obtain the final shape by cutting along the dashed north-east path of the right rectangle. The final shape is symmetric about the double-arrowed line ℓ.

6. (⋆⋆⋆⋆) Continuing with the previous question, Leo performs the operation twice, once with two blue pieces of paper and once with two grey pieces of paper. The grey shape is flipped over and placed

over the blue shape so that the west and north edge of the blue shape aligns with the north and west edges of the grey shape (before flipping), respectively; see Figure 3. Show that Leo can see at least $2r(b-a-r)$ blue unit squares where r is the residue of a modulo $b-a$ (in $\{0,1,\ldots,b-a-1\}$). (Can Leo always cut the papers in such a way to see exactly $2r(b-a-r)$ blue unit squares?)

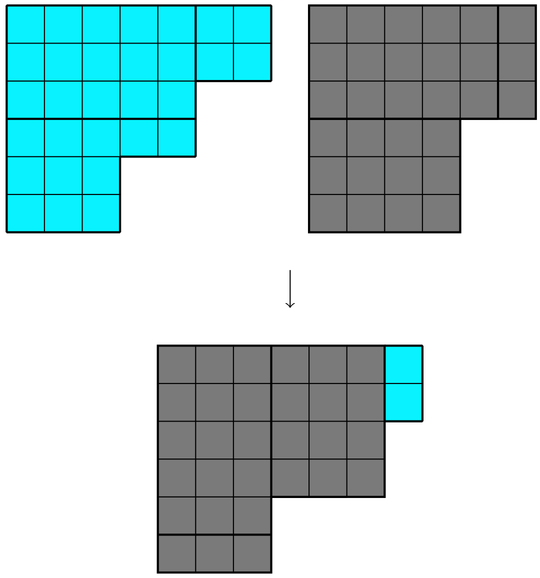

FIGURE 3. An example of two shapes Leo can obtain for $a=3$ and $b=5$. After flipping the grey shape and putting it over the blue shape (so that their north and west edges align), Leo sees precisely two blue unit squares.

7. ($\star$) Let $a,b>1$ be two integers. Charlie has a rectangular $a\times b$ grid of unit squares with rows labelled by $i=1,2,\ldots,a$ from top to bottom and columns labelled by $j=1,2,\ldots,b$ from left to right. Charlie picks some of the unit squares (at least two) and writes in blue colour the number $j+a+1-i$ inside it, where i and j are the row and column of the given square, respectively. Charlie chooses these squares such that for each chosen square, all the squares to its left and above it are also chosen. Charlie's friend Violet then writes in black inside each chosen square S the number of chosen squares on its right and below it (counting S exactly once); see the first grid in Figure 4.

 Charlie and Violet then replace any written number n by $a+b+2-n$, whenever it makes the number smaller; see the second grid in Figure 4. After erasing one black 1 and one blue number, each number appears the same number of times in blue as in black. Show that Charlie has either chosen all boxes, or exactly the boxes in the first row, or exactly the boxes in the first column, and that all these three choices are possible. (How does the answer change if a or b can equal one?)

4 6	5 4	6 3	7 2		
3 5	4 3	5 2	6 1		
2 1					

$\longrightarrow$

4 5	5 4	5 3	4 2		
3 5	4 3	5 2	5 1		
2 1					

FIGURE 4. After Charlie and Violet write their first series of numbers, the rectangular 3×6 grid may look like the one on the left. After the replacement procedure $n\mapsto 11-n$, they obtain the second (final) filling.

REFERENCES TO SOLUTIONS

1. [4, Lemma 6.11 in v2].
2. [4, Lemma 2.9 + Remark 2.10 in v2]. They are more general, so it may be easier to look at a concrete construction in [4, Example 2.8].

3. [4, Lemma 6.12 + Lemma 6.13 in v2]. In the latter, one needs to change the final paragraph, taking advantage of d and e being coprime (this is done in v1 of the preprint).
4. [1, Theorem 1.3] for $t = 1$.
5. [2, Proposition 1.3] with a and b swapped.
6. [2, Proposition 6.2] with a and b swapped.
7. [3, Theorem 5.14(iv)] with $l = a$ and $p = a + b + 2$. In the proof it is assumed that the number of chosen boxes is less than p, but the combinatorial argument does not use this.

References

[1] Jan Petr and Pavel Turek. Towards odd-sunflowers: temperate families and lightnings. *J. Combin. Theory Ser. A*, 220:Paper No. 106133, 11, 2026.

[2] Pavel Turek. Multiplicity-free induced characters of symmetric groups. *Trans. Amer. Math. Soc.*, 377(12):8817–8876, 2024.

[3] Pavel Turek. The representation ring of $\mathrm{SL}_2(\mathbb{F}_p)$ and stable modular plethysms of its natural module in characteristic p. *J. Algebra*, 658:686–715, 2024.

[4] Pavel Turek. Mullineux map: d-balanced partitions and d-runner matrices. *arXiv:2504.03864*, 2025.